ΠΑΝΕΠΙΣΤΗΜΙΟ ΙΩΑΝΝΙΝΩΝ

ΤΜΗΜΑ ΜΑΘΗΜΑΤΙΚΩΝ

UNIVERSITY OF IOANNINA

DEPARTMENT OF MATHEMATICS

Number 20 Volume June-December 2009

A gated type multiclass retrial queue with structured batch arrivals, priorities and vacations

Ioannis Dimitriou
Department of Mathematics, University of Ioannina,
45110 Ioannina, Greece.
E-mail: idimit@cc.uoi.gr

August 11, 2009

Abstract

Motivated by a recent work of Falin [8], we consider a new multiclass batch arrival retrial queue accepting n-types of customers who may arrive in the same batch. If at an arrival epoch the server is idle then the customers of the highest priority in the batch form an ordinary queue waiting to be served while the rest of them leave the system and repeat their demand individually after an exponentially distributed time different for each type of customers. On the other hand, if the server is unavailable, then all customers join their corresponding retrial box. Whenever the server, upon a service completion, phases an empty queue departs for a single vacation. Obviously there is a gate in front of server which is opened when the server is idle. When the server is occupied the gate closes and will be opened again upon the server returns from the vacation. An interesting application of the proposed model, in streaming multimedia applications is also presented. For such a system, we obtain in steady state the mean number of customers in the queue and in each retrial box separately.

Keywords: Multiclass retrial queue, structured batch arrivals, priorities, vacations, general services.

1 Intoduction

Retrial queues are characterized by the feature that an arriving customer who finds upon arrival all servers busy, leaves the service area and repeats his demand after a random amount of time. Such kind of queueing systems are widely used to model computer and communications networks where, for example blocked terminals make retrials to receive service from a central processor, retail shopping queues, telephone-switching systems where a blocked subscriber repeats his call until a successful connection is established etc. For a complete survey of past works and applications on retrial queues see Falin and Templeton [7], Artalejo and Gomez-Corral [2], Kulkarni and Liang [13] and Artalejo [1].

Retrial queues with batch arrivals were considered for the first time by Falin [4], who obtain the generating function of the number of customers in the system. A more detailed analysis of the same model was also given by Falin [5] who studied the non stationary regime and the busy period. Recently, several papers have been published on batch arrival retrial queues. As a related works see Langaris and Moutzoukis [14], Dudin and Klimenok [3], Kim et al. [11] and Ke and Chang [10].

Multiclass retrial queues with batch arrivals were consider firstly by Kulkarny [12], in the case of two types of customers, while Falin [6], using a different methodology extend Kulkarni's results in case of more than two types of customers. Later, Grishechkin [9] analyse the same model using the theory of branching process with immigration and obtain the Laplace transforms of queue lengths, the virtual waiting time and the virtual number of retrials. In all above mentioned works, it is assumed that if a batch of primary customers arrives in the system and the server is free, then one of the customers start to be served and the rest of them leave the service area and repeat their demand later and independently of each other. Later Moutzoukis and Langaris [15] extend the above results by considering a multiclass retrial queue with correlated arrivals, accepting n-types of customers with non-preemptive priorities and vacations, where p-classes form ordinary queues and served according to their priority and the rest $n-p$ classes form retrial queues.

Recently Falin [8] investigates a new batch arrival retrial queue which operates as follows. If the server is free at an arrival epoch, then one of the customers starts to be served and the rest of them form an ordinary queue waiting to be served. In contrast, if the server is busy at an arrival epoch then the whole batch of customers join the retrial box.

In this work we generalize and extend the results of Falin [8], by studying a retrial queue with correlated arrivals, accepting n-types of customers, $P_1, ..., P_n$, say. We always assume that at a batch arrival epoch, P_i customers in batch have priority over $P_j,\ j > i$, to occupy the server. More precisely, if at a batch arrival epoch the server is idle then the customers of the highest priority in the batch form an ordinary queue and start to be served, while the rest of them join their corresponding retrial box. On the other hand, if at an arrival instant the server is unavailable then all the customers in batch join their corresponding retrial box. Whenever the server phases an empty ordinary queue, upon a service completion, departs for a single vacation. Upon returning from the vacation the server remains idle awaiting the first arrival either from outside or from a retrial box. Any retrial customer repeats his demand independently to each other after a random amount of time different for each type, and when the server is in the idle mode (that is when the server returns from a single vacation).

Note here that our model is of gated type. When the server is idle the gate opens. When the server is occupied, either by the customers of highest priority in an arriving batch or by a retrial customer, the gate is closed. While the server is working, arriving customers leave the service area and join their retrial box.

Upon the server phases an empty queue after a service completion, departs for a single vacation. The gate remains closed during vacation period and will be opened again upon the server returns from the vacation period.

From the above description it is clear that the presented model generalize and extend the results of Falin, introducing correlated arrivals, many classes of customers, priorities and vacations. Clearly the assumption of correlated arrivals is valid and common in communication systems and computer network technology (see Sidi [17], Sidi and Segall [16], Takahashi and Takagi [18], Takahashi and Shimogawa [19]) where an arriving message (corresponds to a batch in the model) contains several priority packets (classes of customers). The presented model is well suited to model computer network streaming multimedia applications. The normal queue (that is formed when upon a batch arrival the transmission medium is idle) is similar to an 1-persistent carrier sense multiple access (CSMA) system. When the oldest packet in the normal queue detects that the transmission medium is free, transmission begins immediately. Clearly different type of packets requires different transmission time. If communication medium is unavailable upon a message arrival, then the packets are sent to a retrial queue which is analogous to a non-persistent CSMA system. The retrial packets are retransmiting after a random amount of time (different for each type of packets) before checking the status of the medium again. This procedure is repeated until the retrial packet finds the transmission medium idle.

The presented system can be used to model streaming voice or video in multimedia applications, where transmitted packets are used for playback upon reception and also stored for future use. Arriving messages are consisted of packets that are indexed according to their importance for immediate playback. The packets of the smallest index in the message are those of highest priority and are used for immediate playback. Furthermore the packets of smallest index in batch are time sensitive in that, if they are not transmitted within a given time threshold, they are effectively useless. If the arriving message detects the communication medium idle, then the packets of the smallest index (highest priority) are buffered and transmitted immediately one by one. These packets corresponds to the priority customers. The rest packets in the message, can still be used for later playback from the stored copy of the stream, but their transmission time is no longer important. Moreover, arriving messages that detects the medium unavailable are also used for later playback, while the retransmitting time depends on the type of packet. These packets correspond to the retrial packets. In addition, a close down for a check (vacation) of the medium is necessary, either just after the end of the transmission of the priority packets (packets that need immediate playback), or after the transmission of a retrial packet. Note here that due to the time sensitivity of the priority packets, the check of the communication medium begins after the transmission of all these packets. On the other hand after the transmission of a retrial packet (from the stored copy of a stream) the medium needs immediately a check.

The article is organized as follows. A full description of the model is given in Section 2. In Section 3 we give some useful preliminary results and a theorem on which the whole analysis is based. The steady state analysis of the system is given in Section 4, while the server state probabilities, the mean number of customers in ordinary queue and the mean number of customers in each retrial box separately, are obtained in Section 5.

2 The model

Consider a single server queueing system where customers arrive in batches of random size according to *Poisson* process with parameter λ. Moreover, each batch may contain customers of different types, P_i, $i = 1, 2, ..., n$, say. We assume that at the instant of batch arrival, P_j customers have always priority to occupy first the server, over P_i customers, $j < i$, $\forall\, i, j = 1, ..., n$, that also are included in the arriving batch. Let K_i, $i = 1, ..., n$, the number of P_i customers in an arbitrary batch and let also

$$g(\underline{x}_1) = \Pr(\underline{X}_1 = \underline{x}_1) = \Pr(X_1 = x_n, ..., X_1 = x_n), \qquad g(\underline{0}_1) = 0,$$

$$G(\underline{z}_1) = \sum_{\underline{x}_1=\underline{0}_1}^{\infty} g(\underline{x}_1)\underline{z}_1^{\underline{x}_1} = \sum_{x_1=0}^{\infty} \cdots \sum_{x_n=0}^{\infty} g(x_1, ..., x_n) z_1^{x_1} ... z_n^{x_n},$$

$$g_i = \frac{\partial G(\underline{z}_1)}{\partial z_i}\Big|_{\underline{z}_1=\underline{1}_1}, \quad g_{ki} = \frac{\partial^2 G(\underline{z}_1)}{\partial z_k \partial z_i}\Big|_{\underline{z}_1=\underline{1}_1}.$$

where in general for $i = 1, ..., n$, $\underline{0}_1 = (0, 0, ..., 0)$, $\underline{1}_i = (0, ..., 1, ...0)$,

$$\underline{x}_i = (x_i, x_{i+1}, ..., x_n), \qquad \underline{x}_i^* = (0, ..., 0, x_i, x_{i+1}, ..., x_n),$$
$$\underline{x}_i^{*\ y} = (0, ..., 0, y, x_{i+1}, ..., x_n), \qquad {}_k\underline{x}_i = (x_i, x_{i+1}, ..., x_k), \quad k \geq i.$$

If an arriving batch finds the server idle, then the customers of the highest priority in the batch, P_i, $i = 1, ..., n$, say, form an ordinary queue waiting to be served, while the rest P_j, $j > i$, leave the system and join the jth retrial box from where seek for service individually and independently to the other customers, after an exponentially distributed time with parameter α_j.

On the other hand, if the server is unavailable at the arrival epoch, then the whole batch leave the system and the customers join their corresponding retrial box.

Whenever the server becomes free, that is when upon a service completion there are no customers waiting in the queue (the retrial boxes are not necessary idle), he departs for a single vacation, the length of which is arbitrarily distributed with distribution function (DF) $B_0(x)$, probability density function (pdf) $b_0(x)$, Laplace-Stieltjes Transform (LST) $\beta_0(s)$, finite mean values $\bar{b}_0$ and mth moments $\bar{b}_0^{(m)}$.When the server returns from the vacation remains idle and so available to serve the next arriving customer, either from outside or from a retrial box. It is clear that the server's idle period starts when the server returns from the vacation. Moreover any retrial customer can find a position for service, only when the server is in the idle mode.

From the above description it is clear that when the server is, either busy, or on vacation, any new arriving customers join directly their corresponding retrial box. Thus the presented model is of gated type. The gate opens, whenever the server becomes idle. When the server is occupied, either by the customers of highest priority in an arriving batch or by a retrial customer, the gate is closed and the service area is not accesible for any new arriving customer. Upon the server returns from the single vacation the gate will be opened and the server will become available again.

The service time S_i, of a retrial P_i customer (the customers who upon arrival find the server unavailable), $i = 1, 2, ..., n$, is distributed according to an arbitrarily distribution with DF $B_i(x)$, pdf $b_i(x)$, LST $\beta_i(s)$, finite mean value $\bar{b}_i$ and mth moments $\bar{b}_i^{(m)}$. We also assume that the service time S_i' for the P_i customers, $i = 1, 2, ..., n$, who upon arrival join the queue, is arbitrarily distributed with DF $U_i(x)$, pdf $u_i(x)$, LST $f_i(s)$, finite mean value $\bar{u}_i$ and mth moments $\bar{u}_i^{(m)}$. Finally, all the above defined random variables are assumed to be independent.

3 Preliminary results

In this section we obtain some preliminary results and state a theorem, which is important for the analysis that follows.

Let $A_i(t)$, $i = 1, ..., n$ to be the number of P_i customers that arrive in the interval $(0, t)$ and define

$$s_{\underline{k}_1}(t) = \Pr[A_i(t) = k_i, \quad i = 1, ..., n],$$

then it is easy to understand that

$$\sum_{\underline{k}_1=\underline{0}_1}^{\infty} s_{\underline{k}_1}(t) \underline{z}_1^{\underline{k}_1} = e^{-\lambda(1-G(\underline{z}_1))t}.$$

The generalized completion time of a retrial P_i customer, $i = 1, ..., n$, is defined as the time elapsed from the epoch at which he commences service until the epoch at which the server is idle for the first time. Clearly from the model description, the generalized completion time of a retrial P_i customer equals his service time plus the vacation period that follows. Note that due to the gated type of the model, the server remains always idle upon returning from a vacation.

Let us define by $d_{\underline{k}_1}^{(i)}(t)$ the pdf of such a generalized completion time during which k_r, $r = 1, ..., n$ new P_r customers arrive in the system. Then it is easy to understand that

$$D_i(s, \underline{z}_1) = \sum_{\underline{k}_1=\underline{0}_1}^{\infty} \int_{t=0}^{\infty} e^{-st} d_{\underline{k}_1}^{(i)}(t) \underline{z}_1^{\underline{k}_1} = \beta_i(s + \lambda - \lambda G(\underline{z}_1))\beta_0(s + \lambda - \lambda G(\underline{z}_1)). \tag{1}$$

Define for $i = 1, ..., n$,

$$\rho_i = \lambda g_i \bar{b}_i, \qquad \rho_{0i} = \lambda g_i \bar{b}_0.$$

Now we are ready to state the following theorem. The proof of the Theorem 1 is based on the concept of the generalized completion time and is similar to the proof of Moutzoukis and Langaris [15] and it is ommited here.

Theorem 1 *For any permutation $(i_1, i_2, ..., i_n)$ of the set $(1, 2, ..., n)$ and for (a) $|z_{i_m}| < 1$ for any specific $m = j+1, ..., n$ and $|z_{i_r}| \le 1$ for all other $r = j+1, ..., n$ with $r \neq m$, or (b) $|z_{i_r}| \le 1$ for all $r = j+1, ..., n$ and $\rho^*_{i_{j-1}} > 1$, or (c) $|z_{i_r}| \le 1$ for all $r = j+1, ..., n$ and $\rho^*_{i_j} > 1 \ge \rho^*_{i_{j-1}}$, the equation*

$$z_{i_j} - D_{i_j}(0, w_{i_{j-1}}(z_{i_j}, ..., z_{i_n})) = 0, \tag{2}$$

has, for $j = 1, ..., n$ one and only one root, $z_{i_j} = x_{i_j}(z_{i_{j+1}}, ..., z_{i_n})$, $j \neq n$, $z_{i_n} = x_{i_n}$ say, inside the unit disc $|z_{i_j}| \le 1$, where the vector $w_{i_j}(z_{i_{j+1}}, ..., z_{i_n})$ is defined by

$$\begin{aligned} w_{i_0}(z_{i_1}, ..., z_{i_n}) &= (z_1, ..., z_n), \\ w_{i_1}(z_{i_2}, ..., z_{i_n}) &= w_{i_0}(x_{i_1}(z_{i_2}, ..., z_{i_n}), z_{i_2}, ..., z_{i_n}), \\ w_{i_k}(z_{i_{k+1}}, ..., z_{i_n}) &= w_{i_{k-1}}(x_{i_k}(z_{i_{k+1}}, ..., z_{i_n}), z_{i_{k+1}}, ..., z_{i_n}), \quad k = 1, ..., n-1, \end{aligned}$$

while

$$\rho^*_{i_j} = \textstyle\sum_{m=i_1}^{i_j}(\rho_m + \rho_{0m}).$$

Moreover for $z_{i_r} = 1$, $r = j+1, ..., n$ and $\rho^*_{i_{j-1}} \le 1$ the root $x_{i_j}(1, ..., 1)$ is the smallest positive real root of (2) with $x_{i_j}(1, ..., 1) < 1$ if $\rho^*_{i_j} > 1$ and $x_{i_j}(1, ..., 1) = 1$ if $\rho^*_{i_j} \le 1$.

By differentiating both sides of (1), with respect to z_i, at the point $s = 0$, $\underline{z}_1 = \underline{1}_1$ we obtained for $i = 1, ..., n$

$$\bar{\rho}_i = \rho_i + \rho_{0i},$$

which is the traffic intensity of the retrial P_i customers.

Thus the total traffic intensity is given by

$$\rho^* = \textstyle\sum_{i=1}^n(\rho_i + \rho_{0i}) = \lambda \sum_{i=1}^n g_i(\bar{b}_i + \bar{b}_0).$$

In the following sections we shall consider the system in steady state, which exists if and only if $\rho^* < 1$. Thus the condition $\rho^* < 1$ is assumed to hold from hear on. Note here that $\rho_i + \rho_{0i}$ represents the expected number of retrial P_i customers that arrive during the generalized completion time of a retrial P_i customer.

4 Steady state analysis

Let us assume that assume that the system is in steady state, so that $\rho^* < 1$. Let also Q_i, $i = 1, ..., n$, be the number of P_i customers in the queue, and N_i, $i = 1, ..., n$, be the number of P_i customers in the ith retrial box. Define finally

$$\xi = \begin{cases} u_i, & \text{busy with a } P_i \text{ customer, } i = 1, ..., n \text{ from the queue,} \\ b_i, & \text{busy with a } P_i \text{ customer, } i = 1, ..., n \text{ from the } i\text{th retrial box,} \\ 0, & \text{vacation,} \\ id, & \text{idle,} \end{cases}$$

the random variable that describes server's state, and let for $i = 1, ..., n$,

$$p_i'(m_i, \underline{k}_1, x) = \Pr(Q_i = m_i, \underline{N}_1 = \underline{k}_1, \xi = u_i, x < \bar{U}_i \le x + dx), \tag{3}$$

and

$$\begin{aligned} p_i(\underline{k}_1, x) &= \Pr(\underline{N}_1 = \underline{k}_1, \xi = b_i, x < \bar{B}_i \le x + dx), \qquad i = 1, ..., n, \\ p_0(\underline{k}_1, x) &= \Pr(\underline{N}_1 = \underline{k}_1, \xi = 0, x < \bar{B}_0 \le x + dx), \\ q(\underline{k}_1) &= \Pr(\underline{N}_1 = \underline{k}_1, \xi = id), \end{aligned} \tag{4}$$

where $\bar{W}_i$ is the elapsed time period of any random variable W. Define also for $i = 1, ..., n$,

$$\begin{aligned} P_i'(y_i, \underline{z}_1, x) &= \sum_{m_i=0}^{\infty} \sum_{\underline{k}_1=\underline{0}_1}^{\infty} p_i'(m_i, \underline{k}_1, x) y_i^{m_i} \underline{z}_1^{\underline{k}_1} \\ P_i(\underline{z}_1, x) &= \sum_{\underline{k}_1=\underline{0}_1}^{\infty} p_i(\underline{k}_1, x) \underline{z}_1^{\underline{k}_1}, \\ Q(\underline{z}_1) &= \sum_{\underline{k}_1=\underline{0}_1}^{\infty} q_i(\underline{k}_1) \underline{z}_1^{\underline{k}_1}. \end{aligned}$$

By connecting as usual the probabilities (3), (4) to each other and forming the above defined generating functions we obtain for $i = 1, ..., n$,

$$\begin{aligned} P_i'(y_i, \underline{z}_1, x) &= P_i'(y_i, \underline{z}_1, 0)(1 - U_i(x)) \exp[-(\lambda - \lambda G(\underline{z}_1))x], \\ P_i(\underline{z}_1, x) &= P_i(\underline{z}_1, 0)(1 - B_i(x)) \exp[-(\lambda - \lambda G(\underline{z}_1))x], \end{aligned} \tag{5}$$

$$\lambda Q(\underline{z}_1) + \sum_{i=1}^{n} \alpha_i z_i \frac{\partial}{\partial z_i} Q(\underline{z}_1) = \int_0^{\infty} P_0(\underline{z}_1,) \eta_0(x) dx, \tag{6}$$

where $\eta_j(x) = b_j(x)/(1 - B_j(x))$, $\eta_j'(x) = u_j(x)/(1 - U_j(x))$. Let us define $k_i(\underline{z}_1) = \beta_i(\lambda - \lambda G(\underline{z}_1))$, $r_i(\underline{z}_1) = f_i(\lambda - \lambda G(\underline{z}_1))$, $i = 1, ..., n$.

The corresponding boundary conditions are given by

$$\begin{aligned} p_i'(m_i, \underline{k}_1, 0) &= \lambda \sum_{\underline{t}_{i+1}=\underline{0}_{i+1}}^{\underline{k}_{i+1}} g(\underline{t}_i^{*\ m_i+1}) q({}_i\underline{k}_1, \underline{k}_{i+1} - \underline{t}_{i+1}) \\ &\quad \int_0^{\infty} p_i'(m_i + 1, \underline{k}_1, x) \eta_i'(x) dx, \quad i = 1, 2, ..., n, \\ p_i(\underline{k}_1, 0) &= \alpha_i (k_i + 1) q(\underline{k}_1 + \underline{1}_i), \quad i = 1, ..., n \\ p_0(\underline{k}_1, 0) &= \sum_{i=1}^{n} [\int_0^{\infty} p_i(\underline{k}_1, x) \eta_i(x) dx + \int_0^{\infty} p_i'(0, \underline{k}_1, x) \eta_i'(x) dx]. \end{aligned}$$

Forming the generating functions, we arrive easily for $i = 1, ..., n$,

$$(y_i - r_i(\underline{z}_1)) P_i'(y_i, \underline{z}_1, 0) = \lambda [G(\underline{z}_i^{*\ y_i}) - G(\underline{z}_{i+1}^*)] Q(\underline{z}_1) - P_i'(0, \underline{z}_1, 0) r_i(\underline{z}_1). \tag{7}$$

Then setting, $y_i = r_i(\underline{z}_1)$, we obtain

$$P_i'(y_i, \underline{z}_1, 0) = \frac{\lambda [G(\underline{z}_i^{*\ y_i}) - G(\underline{z}_i^{*\ r_i(\underline{z}_1)})] Q(\underline{z}_1)}{y_i - r_i(\underline{z}_1)}. \tag{8}$$

Moreover

$$\begin{aligned} P_i(\underline{z}_1, 0) &= \alpha_i z_i \frac{\partial}{\partial z_i} Q(\underline{z}_1), \\ P_0(\underline{z}_1, 0) &= \sum_{i=1}^{n} P_i(\underline{z}_1, 0) k_i(\underline{z}_1) + \sum_{i=1}^{n} P_i'(0, \underline{z}_1, 0) r_i(\underline{z}_1). \end{aligned} \tag{9}$$

Integrating (5) with respect to x, we realise that $P_{.}(\underline{z}_1, 0) = P_{.}(\underline{z}_1)e_{.}(\underline{z}_1)$, $P_{.}^{'}(y_{.}, \underline{z}_1, 0) = P_{.}^{'}(y_{.}, \underline{z}_1)h_{.}(\underline{z}_1)$, where

$$e_{.}(\underline{z}_1) = \frac{\lambda-\lambda G(\underline{z}_1)}{1-k_{.}(\underline{z}_1)} \qquad h_{.}(\underline{z}_1) = \frac{\lambda-\lambda G(\underline{z}_1)}{1-r_{.}(\underline{z}_1)}.$$

Then relations (8), (9), becomes

$$h_i(\underline{z}_1)P_i^{'}(y_i, \underline{z}_1) = \frac{\lambda[G(\underline{z}_i^{*\ y_i}) - G(\underline{z}_i^{*\ r_i(\underline{z}_1)})]Q(\underline{z}_1)}{y_i - r_i(\underline{z}_1)}. \tag{10}$$

$$\begin{aligned} e_i(\underline{z}_1)P_i(\underline{z}_1) = &\ \alpha_i z_i \tfrac{\partial}{\partial z_i} Q(\underline{z}_1), \\ e_0(\underline{z}_1)P_0(\underline{z}_1) = &\ \textstyle\sum_{i=1}^{n} e_i(\underline{z}_1)P_i(\underline{z}_1)k_i(\underline{z}_1) \\ &+\lambda Q(\underline{z}_1)\textstyle\sum_{i=1}^{n}[G(\underline{z}_i^{*\ r_i(\underline{z}_1)}) - G(\underline{z}_{i+1}^{*})]. \end{aligned} \tag{11}$$

Moreover (6) becomes

$$\lambda Q(\underline{z}_1) + \textstyle\sum_{i=1}^{n} \alpha_i z_i \tfrac{\partial}{\partial z_i} Q(\underline{z}_1) = e_0(\underline{z}_1)P_0(\underline{z}_1)k_0(\underline{z}_1). \tag{12}$$

Note here that if we exclude the concept of vacations, assume that any batch contains only one class of customers and that the service time $S \equiv S^{'}$, then the above defined generating functions $Q(\underline{z}_1)$, $P_i(\underline{z}_1)$, $P_i^{'}(y_i, \underline{z}_1)$ becomes $Q(z)$, $P(z)$, $P^{'}(y, z)$ respectively. In that case our model yield to that of Falin [8]. More precisely, the generating function of the total number of customers in the system in page 4 in Falin [8] ($P(y, z)$), is connected with the corresponding in our paper $Q(z)$, $P(z)$, $P^{'}(y, z)$ with the following relation,

$$P(y, z) = Q(z) + y[P(z) + P^{'}(y, z)].$$

Substituting (11) to (12), we arrive after manipulations to the n-dimensional partial differential equation

$$\textstyle\sum_{i=1}^{n} \alpha_i[z_i - D_i(0, \underline{z}_1)]\tfrac{\partial}{\partial z_i} Q(\underline{z}_1) + \lambda[1 - k_0(\underline{z}_1)F(\underline{z}_1)]Q(\underline{z}_1) = 0, \tag{13}$$

where

$$\begin{aligned} D_i(0, \underline{z}_1) = &\ k_i(\underline{z}_1)k_0(\underline{z}_1), \quad i = 1, ..., n, \\ F(\underline{z}_1) = &\ \textstyle\sum_{i=1}^{n}[G(\underline{z}_i^{*\ r_i(\underline{z}_1)}) - G(\underline{z}_{i+1}^{*})]. \end{aligned}$$

It is clear that, in order to obtain the generating functions (10)-(11) we have to solve first equation (13), which hardly can be solved. Our objective now is to investigate the mean number of customers both in priority queue and in each retrial box separately, by using the relations obtain so far and a special methodology used first in Falin [6].

5 Performance measures

To proceed to the main analysis we have to calculate at point $y_i = 1,\ \underline{z}_1 = \underline{1}_1$ the generating functions (10)-(11).

Theorem 2 *For $\rho^* < 1$, the generating functions (10)-(11), at point $y_i = 1,\ \underline{z}_1 = \underline{1}_1$ are given by*

$$\begin{aligned} P_i(\underline{1}_1) =& \quad \lambda\bar{b}_i(g_i - g_i^* Q(\underline{1}_1)), \quad i = 1, ..., n, \\ P_i^{'}(1, \underline{1}_1) =& \quad \lambda g_i^* \bar{u}_i Q(\underline{1}_1), \quad i = 1, ..., n, \\ P_0(\underline{1}_1) =& \quad \lambda\bar{b}_0\{\textstyle\sum_{i=1}^n g_i + Q(\underline{1}_1)[1 - \sum_{i=1}^n g_i^*]\}, \\ Q(\underline{1}_1) =& \quad \frac{1-\rho^*}{1+\lambda\bar{b}_0+\sum_{i=1}^n \lambda g_i^*[\bar{u}_i - \bar{b}_i - \bar{b}_0]}. \end{aligned} \tag{14}$$

where $g_i^* = \frac{\partial G(z_i^*)}{\partial z_i}|_{z_i^* = 1_i^*}$.

Proof: Let us define

$$\mathcal{N}(\underline{y}_1, \underline{z}_1) = Q(\underline{z}_1) + \textstyle\sum_{i=0}^n P_i(\underline{z}_1) + \sum_{i=1}^n P_i^{'}(y_i, \underline{z}_1). \tag{15}$$

Setting in (10) $y_i = 1,\ \underline{z}_1 = \underline{1}_1$ we obtain easily

$$P_i^{'}(1, \underline{1}_1) = \lambda g_i^* \bar{u}_i Q(\underline{1}_1), \quad i = 1, ..., n,$$

while the second of (11) becomes

$$P_0(\underline{1}_1) = \bar{b}_0 \textstyle\sum_{i=1}^n \frac{P_i(\underline{1}_1)}{\bar{b}_i} + \lambda\bar{b}_0 Q(\underline{1}_1). \tag{16}$$

Substituting the first of (11) to (13) we easily arrive at,

$$\frac{[1 - k_0(\underline{z}_1)F(\underline{z}_1)]Q(\underline{z}_1)}{1 - G(\underline{z}_1)} = \sum_{i=1}^n \frac{P_i(\underline{z}_1)}{1 - k_i(\underline{z}_1)}[D_i(0, \underline{z}_1) - z_i] \tag{17}$$

Consider now any arbitrary permutation $(i_1, ..., i_n)$ of the set $(1, 2, ..., n)$. Then using Theorem 1 and replacing repeatedly in (17) z_{i_j} the corresponding root $x_{i_j}(\underline{z}_{i_{j+1}})$ for $j = 1, ..., n-1$, we could eliminate all except one term of the left hand side of (17). After manipulations we arrine for $i = 1, ..., n$ in

$$P_i(\underline{1}_1) = \quad \frac{\lambda\bar{b}_i}{1-\rho} Q(\underline{1}_1)\{g_i[1 + \lambda\bar{b}_0 + \textstyle\sum_{j=1}^n \lambda g_j^*(\bar{u}_j - \bar{b}_j - \bar{b}_0)] + g_i^*(\rho^* - 1)\} \tag{18}$$

Substituting (18) in (16) and using the fact that $\mathcal{N}(\underline{1}_1, \underline{1}_1) = 1$, we obtain after some algebra

$$Q(\underline{1}_1) = \frac{1 - \rho^*}{1 + \lambda\bar{b}_0 + \sum_{i=1}^n \lambda g_i^*[\bar{u}_i - \bar{b}_i - \bar{b}_0]}. \tag{19}$$

Replacing (19) in (18) and (16), we arrive at the first and third of (14) respectively. □

Theorem 3 *The mean number $\bar{Q}_k$, of P_k, $k = 1, .., n$ customers in the queue is given by*

$$\bar{Q}_k = \frac{\lambda \bar{u}_k Q(\underline{1}_1)}{2} \times \frac{\partial^2 G(\underline{1}_k^{*\ y_k})}{\partial y_k^2}|_{y_k=1}. \tag{20}$$

Proof: Setting $\underline{z}_1 = \underline{1}_1$ in relation (10) we arrive at

$$P_k'(y_k, \underline{1}_1) = \lambda \bar{u}_k Q(\underline{1}_1) \frac{[G(\underline{1}_k^{*\ y_k}) - G(\underline{1}_k^*)]}{y_k - 1}.$$

Differentiating at point $y_k = 1$ we easily obtain relation (20) and the theorem has been proved. □

Define now for every $k = 1, ..., n$,

$$\begin{aligned}\pi_k = \quad & \tfrac{\partial Q(\underline{z}_1)}{\partial z_k}|_{\underline{z}_1=\underline{1}_1}\{1 + (\lambda + \alpha_k)\bar{b}_0 + \textstyle\sum_{i=1}^n \lambda g_i^* \bar{u}_i - \lambda \sum_{i=1}^n \theta_{ki} g_i \\ & +\lambda \textstyle\sum_{i=2}^n \theta_{ki}(g_i - g_i^*)\} - \lambda(g_k - \delta_{\{k>1\}}(g_k - g_k^*)) \sum_{i=1}^n \theta_{ki} \tfrac{\partial Q(\underline{z}_1)}{\partial z_i}|_{\underline{z}_1=\underline{1}_1} \\ & +\lambda Q(\underline{1}_1)\delta_{\{k>1\}}\{\textstyle\sum_{i=2}^n \theta_{ki} g_{ki} - \sum_{i=k+1}^n \theta_{ki} \tfrac{\partial^2 G(\underline{z}_k^*)}{\partial z_k \partial z_i}|_{\underline{z}_k^*=\underline{1}_k^*} \\ & -\textstyle\sum_{i=2}^{k-1} \theta_{ki} \tfrac{\partial^2 G(\underline{z}_i^*)}{\partial z_k \partial z_i}|_{\underline{z}_i^*=\underline{1}_i^*} - \theta_{kk} \tfrac{\partial^2 G(\underline{z}_k^*)}{\partial z_k^2}|_{\underline{z}_k^*=\underline{1}_k^*}\} \\ & +\lambda g_k\{\tfrac{1}{2} \textstyle\sum_{i=1}^n [g_i^* \bar{u}_i^{(2)} + \bar{b}_i^{(2)}(g_i - g_i^*)Q(\underline{1}_1)] + P_0(\underline{1}_1)[\tfrac{\bar{b}_0^{(2)}}{2\bar{b}_0} - \bar{b}_0]\},\end{aligned} \tag{21}$$

where $\theta_{ki} = \alpha_i(\bar{b}_i + \bar{b}_0)/(\alpha_k + \alpha_i)$.

The next step is to obtain the mean number of P_k, $k = 1, ..., n$ customers in each retrial box separately.

Theorem 4 *The mean number $\bar{N}_k$ of P_k, $k = 1, ..., n$ customers in each retrial box, can be found as a sulution of the following system of linear equations*

$$[1 - \lambda \textstyle\sum_{i=1}^n \theta_{ki} g_i]\bar{N}_k - \lambda g_k \sum_{i=1}^n \theta_{ki} \bar{N}_i = \quad \lambda(1 - Q(\underline{1}_1)) \sum_{i=1}^n \theta_{ik} g_{ki} + \pi_k, \tag{22}$$

where π_k is given by (21).

Proof: It is clear that

$$\bar{N}_k = \frac{\partial Q(\underline{z}_1)}{\partial z_k}|_{\underline{z}_1=\underline{1}_1} + \sum_{i=0}^n \frac{\partial P_i(\underline{z}_1)}{\partial z_k}|_{\underline{z}_1=\underline{1}_1} + \sum_{i=1}^n \frac{\partial P_i'(1, \underline{z}_1)}{\partial z_k}|_{\underline{z}_1=\underline{1}_1}. \tag{23}$$

Differentiating now the relations (10), (11), we obtain $\partial P_i/\partial z_k$, $\partial P_i'/\partial z_k$ as functions of $\partial Q/\partial z_k$ and $\partial^2 Q/\partial z_k \partial z_i$. Clearly, by differentiating relation (10),

(11) we obtain

$$
\begin{aligned}
\frac{\partial P_i'(1,\underline{z}_1)}{\partial z_k}|_{\underline{z}_1=\underline{1}_1} &= \lambda g_i^* \bar{u}_i \frac{\partial Q(\underline{z}_1)}{\partial z_k}|_{\underline{z}_1=\underline{1}_1} + \frac{\lambda g_k g_i^*}{2}\bar{u}_i^{(2)}, \\
\frac{\partial P_i(\underline{z}_1)}{\partial z_k}|_{\underline{z}_1=\underline{1}_1} &= \alpha_i \bar{b}_i \frac{\partial^2 Q(\underline{z}_1)}{\partial z_k \partial z_i}|_{\underline{z}_1=\underline{1}_1} + \frac{\lambda^2 g_k \bar{b}_i^{(2)}(g_i - g_i^* Q(\underline{1}_1))}{2}, \quad i=1,...,n \\
\frac{\partial P_0(\underline{z}_1)}{\partial z_k}|_{\underline{z}_1=\underline{1}_1} &= \sum_{i=1}^n \alpha_i \bar{b}_0 \frac{\partial^2 Q(\underline{z}_1)}{\partial z_k \partial z_i}|_{\underline{z}_1=\underline{1}_1} + (\lambda+\alpha_k)\bar{b}_0 \frac{\partial Q(\underline{z}_1)}{\partial z_k}|_{\underline{z}_1=\underline{1}_1} \\
&\quad + \lambda g_k P_0(\underline{1}_1)[\frac{\bar{b}_0^{(2)}}{2\bar{b}_0} - \bar{b}_0].
\end{aligned}
\tag{24}
$$

The main problem now, is to obtain a formula for $\partial^2 Q/\partial z_k \partial z_i$.

Following the methodology of Falin [6] and using relations (10)-(12) we arrive after manipulations at the following basic equation

$$
\begin{aligned}
\lambda Q(\underline{z}_1)[G(\underline{z}_1) - \textstyle\sum_{i=1}^n (G(\underline{z}_i^{*\,1}) - G(\underline{z}_{i+1}^*))] \\
+ \textstyle\sum_{i=1}^n \alpha_i (z_i - 1)\frac{\partial Q(\underline{z}_1)}{\partial z_i} \quad &= -(\lambda - \lambda G(\underline{z}_1))\mathcal{N}(\underline{1}_1, \underline{z}_1).
\end{aligned}
\tag{25}
$$

Differentiate (25) twice, firstly with respect of z_k and then with respect of z_i, setting finally $\underline{z}_1 = \underline{1}_1$ we obtain after some algebra

$$
\begin{aligned}
(\alpha_k + \alpha_i)\frac{\partial^2 Q(\underline{z}_1)}{\partial z_k \partial z_i}|_{\underline{z}_1=\underline{1}_1} = \; & \lambda(g_k \bar{N}_i + g_i \bar{N}_k) + \lambda g_{ki}(1 - Q(\underline{1}_1)) - \lambda \frac{\partial Q(\underline{z}_1)}{\partial z_k}|_{\underline{z}_1=\underline{1}_1} \\
& \times [g_i - \delta_{\{i>1\}}(g_i - g_i^*)] - \lambda \frac{\partial Q(\underline{z}_1)}{\partial z_i}|_{\underline{z}_1=\underline{1}_1}[g_k \\
& - \delta_{\{k>1\}}(g_k - g_k^*)] + \lambda Q(\underline{1}_1)\delta_{\{k,i>1\}}[g_{ki} \\
& - \delta_{\{i>k\}}\frac{\partial^2 G(\underline{z}_k^*)}{\partial z_k \partial z_i}|_{\underline{z}_k^*=\underline{1}_k^*} - \delta_{\{k>i\}}\frac{\partial^2 G(\underline{z}_i^*)}{\partial z_k \partial z_i}|_{\underline{z}_i^*=\underline{1}_i^*} \\
& - \delta_{\{k=i\}}\frac{\partial^2 G(\underline{z}_i^*)}{\partial z_i^2}|_{\underline{z}_i^*=\underline{1}_i^*}],
\end{aligned}
\tag{26}
$$

where

$$
\delta_{\{A\}} = \begin{cases} 1, & \text{if } A \text{ holds,} \\ 0, & \text{else.} \end{cases}
$$

Replacing finally (24) using (26), in (23) we arrive after manipulations at (22) and this proves the theorem. □

6 Conclusion

In this work we study a new multiclass retrial queue with structured batch arrivals, priorities and vacations, motivated by a recent work of Falin [8]. If an arriving batch finds the server idle, then the customers of the highest priority in batch form an ordinary queue, while the rest customers join their corresponding retrial box. In contrast, if the server is unavailable at the epoch of a batch arrival, all the customers in batch join their corresponding retrial box. Retrial customers seek for service individually and independently after a random amount of time, different for each type of customers. Upon a service completion, if the server phases an empty ordinary queue, he departs for a single vacation. Upon the server returns from the vacation remains idle awaiting the first arrival, either from outside or from a retrial box to start the service procedure again. For such a system the mean number of customers that form an ordinary queue upon a batch arrival are obtained in closed form. Using a special methodology, first used in Falin [6], the mean number of customers in each retrial box are obtained as a solution of a system of linear equations.

References

[1] J.R. Artalejo, A classified bibliography of research on retrial queues: Progress in 1990-1999, Top 7(2) (1999) 187-211.

[2] J.R. Artalejo, A. Gomez-Corral, *Retrial Queueing Systems, A Computational Approach,* Springer-Verlag Berlin Heidelberg (2008).

[3] A.N. Dudin, V.I. Klimenok, A retrial BMAP/SM/1 system with linear repeated requests, Queueing Systems 34 (2000) 47-66.

[4] G.I. Falin, Aggregate arrival of customers in one-line system with repeated calls, Ukrainian Mathematical Journal 28(4) (1976) 437-440.

[5] G.I. Falin, Functioning under nonsteady conditions of a single-channel system with group arrival of requests and repeated calls, Ukrainian Mathematical Journal 33(4) (1981) 429-432.

[6] G.I. Falin, On multiclass batch arrival retrial queue, Advances in Applied Probability 20(2) (1988) 483-487.

[7] G.I. Falin, J.G.C. Templeton, *Retrial Queues*, Chapman and Hall, London (1997).

[8] G.I. Falin, A single-server batch arrival retrial queue with returning customers, European Journal of Operational Research, *In Press* (2009).

[9] S.A. Grieshechkin, Multiclass batch arrival retrial queues analyzed as branching process with immigration, Queueing Systems 11(4) (1992) 395-418.

[10] J.-C. Ke, F.-M. Chang, $M^{[x]}/(G_1, G_2)/1$ retrial queue with under Bernoulli vacation schedules with general repeated attempts and starting failures, Applied Mathematical Modelling 33 (2009) 3186-3196.

[11] C. Kim, V.I. Klimenok, D.S. Orlovsky, The BMAP/PH/N retrial queue with Markovian flow of breakdowns, European Journal of Operational Research 189(3) (2008) 1057-1082.

[12] V.G. Kulkarni, Expected waiting times in a multiclass batch arrival retrial queue, Journal of Applied Probability 23(1) (1986) 144-159.

[13] V.G. Kulkarni, H.M. Liang, Retrial queues revisited, in: Frontiers in Queueing, ed. J.H. Dshalalow, CRP Press, 1997, pp. 19-34.

[14] C. Langaris and Moutzoukis E., A retrial queue with structured batch arrivals, priorities and server vacations, Queueing Systems 20 (1995) 341-368.

[15] Moutzoukis E., C. Langaris, Non-preemptive priorities and vacations in a multiclass batch arrival retrial queueing system, Stochastic Models 12(3) (1996) 455-472.

[16] M. Sidi, A. Segall, Structured Priority Queueing Systems with Applications to Packet-Radio Networks, Performance Evaluation 3 (1983) 265-275.

[17] M. Sidi, Discrete-Time Priority Queues with Partial Interference, IEEE Journal On Selected Areas In Communications, SAC-5(6) (1987) 1041-1050.

[18] Y. Takahashi, H. Takagi, Structured Priority Queue With Batch Arrivals, Journal of the Operations Research of Japan 33(3) (1990) 242-261.

[19] Y. Takahashi, S. Shimogawa, Composite priority single server queue with structured batch inputs, Stochastic Models 7(3) (1991) 481-497.

A fixed point's theorem on three complete metric spaces

Luljeta Kikina *

Department of Mathematics, Faculty of Natural Sciences
University of Gjirokastra
Albania
gjonileta@yahoo.com

Abstract. A fixed point theorem for three mappings on three metric spaces is proved. This result is a modification of the result of Nešic′ [1] from two mappings of a metric space into itself, to three mappings of different metric spaces. We have modified the methods used by Nešic′ [1] and by Jain, Shrivastava and Fischer [3]. We also show that the Theorem of Nung [2] is a corollary of our result and that it is sufficient the continuity of only one of the mappings. An application of our result is presented.

Keywords: fixed point, metric space, complete metric space

1. Introduction

In [1], the following theorem is proved:

Theorem 1.1 *Let (X,d) be a metric space and S,T be two mappings of X into itself, satisfying the following inequality:*

$$[1+pd(x,y)]d(Sx,Ty) \le p[d(x,Sx)d(y,Ty)+d(x,Ty)d(y,Sx)]+$$

$$+q\max\{d(x,y),d(x,Sx),d(y,Ty),\frac{1}{2}[d(x,Ty)+d(y,Sx)]\}$$

for all $x,y\in X$, where $p\ge 0$ and $0\le q<1$.

If (X,d) is (S,T)-orbitally complete metric space, then S and T have an unique common fixed point u in X.

In [2], the following theorem is proved:

Theorem 1.1 *Let $(X,d_1),(Y,d_2),(Z,d_3)$ be three complete metric spaces and T be a continuous mapping of X into Y, S a continuous mapping of Y into Z and R be a continuous of Z into X, satisfying the following inequality:*

$$d_1(RSTx,RSy)\le c\max\{d_1(x,RSy),d_1(x,RSTx),d_2(y,Tx),d_3(Sy,STx)\}$$

$$d_2(TRSy,TRz)\le c\max\{d_2(y,TRz),d_2(y,TRSy),d_3(Z,Sy),d_1(Rz,RSy)\}$$

$$d_3(STRz,STx)\le c\max\{d_3(z,STx),d_3(z,STRz),d_1(x,Rz),d_2(Tx,TRz)\}$$

for all $x\in X, y\in Y$ and $z\in Z$, where $0\le c<1$. Then RST has an unique fixed point $u\in X$, TRS has an unique fixed point $v\in Y$ and STR has an unique fixed point $w\in Z$. Further, $Tu=v, Sv=w$ and $Rw=u$.

---

*This work was done while the author was visiting the Department of Mathematics, University of Ioannina, in the framework of a Post – doctoral scholarship offered by IKY (State Scholarships Foundation), Athens, Greece.

In this paper we will give a generalization of Theorem 1.2 modifying the results of Nešic′ [1]. We will also show that in Theorem 1.2 it is not nescessary the continuity of the three mappings, but it is sufficient the continuity of only one of them.

An application of our result is presented.

2. Main results

Theorem 2.1 *Let* $(X,d_1),(Y,d_2),(Z,d_3)$ *be three complete metric spaces and* $T:X\to Y, S:Y\to Z$ *and* $R:Z\to X$ *be three mappings from which at least one of them is continuous, satisfying the following inequality:*

$$[1 + pd_1(x,RSy)+pd_2(y,Tx)]d_1(RSy,RSTx)\le$$
$$\le p[d_1(x,RSy)d_3(Sy,STx)+d_1(x,RSTx)d_2(y,TRSy)+d_1(x,RSy)d_2(y,Tx)]+$$
$$+\ q\max\{d_1(x,RSy),d_1(x,RSTx),d_2(y,Tx),d_3(STx,Sy)\} \quad (1)$$

$$[1 + pd_2(y,TRz)+pd_3(z,Sy)]d_2(TRz,TRSy)\le$$
$$\le p[d_2(y,TRz)d_1(Rz,RSy)+d_2(y,TRSy)d_3(z,STRz)+d_2(y,TRz)d_3(z,Sy)]+$$
$$+\ q\max\{d_2(y,TRz),d_2(y,TRSy),d_3(z,Sy),d_1(RSy,Rz)\} \quad (2)$$

$$[1 + pd_3(z,STx)+pd_1(x,Rz)]d_3(STx,STRz)\le$$
$$\le p[d_3(z,STx)d_2(Tx,TRz)+d_3(z,STRz)d_1(x,RSTx)+d_3(z,STx)d_1(x,Rz)]+$$
$$+\ q\max\{d_3(z,STx),d_3(z,STRz),d_1(x,Rz),d_2(TRz,Tx)\} \quad (3)$$

for all $x\in X, y\in Y, z\in Z$*, where* $p\ge 0$ *and* $0\le q<1$*. Then* RST *has an unique fixed point* $\alpha\in X$*,* TRS *has an unique fixed point* $\beta\in Y$ *and* STR *has an unique fixed point* $\gamma\in Z$*. Further,* $T\alpha=\beta, S\beta=\gamma$ *and* $R\gamma=\alpha$*.*

Proof. Let $x_0\in X$ be an arbitrary point. We define the sequences $(x_n),(y_n)$ and (z_n) in X,Y and Z respectively as follows:

$$x_n=(RST)^n x_0,\ y_n=Tx_{n-1},\ z_n=Sy_n$$

for $n=1,2,\ldots$

By the inequality (2), for $y=y_n$ and $z=z_{n-1}$ we get:

$$[1 + pd_2(y_n,y_n)+pd_3(z_{n-1},z_n)]d_2(y_n,y_{n+1})\le$$
$$\le p[d_2(y_n,y_n)d_1(x_{n-1},x_n)+d_2(y_n,y_{n+1})d_3(z_{n-1},z_n)+d_2(y_n,y_n)d_3(z_{n-1},z_n)]+$$
$$+\ q\max\{d_2(y_n,y_n),d_2(y_n,y_{n+1}),d_3(z_{n-1},z_n),d_1(x_n,x_{n-1})\}$$

from which it follows:

$$d_2(y_n,y_{n+1})\le q\max\{d_2(y_n,y_{n+1}),d_1(x_n,x_{n-1}),d_3(z_n,z_{n-1})\}=q\max A$$

where $A=\{d_2(y_n,y_{n+1}),d_1(x_n,x_{n-1}),d_3(z_n,z_{n-1})\}$.

If $\max A=d_2(y_n,y_{n+1})$, then we have:

$$d_2(y_n,y_{n+1})\le qd_2(y_n,y_{n+1})$$

and since $0\le q<1$, it follows $d_2(y_n,y_{n+1})=0$.

Thus we have:

$$d_2(y_n, y_{n+1}) \le q \max\{d_1(x_n, x_{n-1}), d_3(z_n, z_{n-1})\} \qquad (4)$$

In the same way, by (3), for $x = x_{n-1}$ and $z = z_n$, we get:

$$[1 + pd_3(z_n, z_n) + pd_1(x_{n-1}, x_n)]d_3(z_n, z_{n+1}) \le$$
$$\le p[d_3(z_n, z_n)d_2(y_n, y_{n+1}) + d_3(z_n, z_{n+1})d_1(x_{n-1}, x_n) + d_3(z_n, z_n)d_1(x_n, x_n)] +$$
$$+ q \max\{d_3(z_n, z_n), d_3(z_n, z_{n+1}), d_1(x_{n-1}, x_n), d_2(y_{n+1}, y_n)\}$$

from which we get:

$$d_3(z_n, z_{n+1}) \le q \max\{d_1(x_{n-1}, x_n), d_3(z_{n-1}, z_n)\} \qquad (5)$$

In the same way, by (1), for $y = y_n$ and $x = x_n$ we get:

$$[1 + pd_1(x_n, x_n) + pd_2(y_n, y_{n+1})]d_1(x_n, x_{n+1}) \le$$
$$\le p[d_1(x_n, x_n)d_3(z_n, z_{n+1}) + d_1(x_n, x_{n+1})d_2(y_n, y_{n+1}) + d_1(x_n, x_n)d_2(y_n, y_{n+1})] +$$
$$+ q \max\{d_1(x_n, x_n), d_1(x_n, x_{n+1}), d_2(y_n, y_{n+1}), d_3(z_{n+1}, z_n)\}$$

from which we get:

$$d_1(x_n, x_{n+1}) \le q \max\{d_1(x_n, x_{n+1}), d_2(y_n, y_{n+1}), d_3(z_n, z_{n+1})\}$$

and by (4) and (5) we have:

$$d_1(x_n, x_{n+1}) \le q \max\{d_1(x_{n-1}, x_n), d_3(z_{n-1}, z_n)\} \qquad (6)$$

Taking n equal with $n-1, n-2, \ldots.$, using (4), (5) and (6) we obtain:

$$d_1(x_n, x_{n+1}) \le q^{n-1} \max\{d_1(x_1, x_2), d_3(z_1, z_2)\}$$
$$d_2(y_n, y_{n+1}) \le q^{n-1} \max\{d_1(x_1, x_2), d_3(z_1, z_2)\}$$
$$d_3(z_n, z_{n+1}) \le q^{n-1} \max\{d_1(x_1, x_2), d_3(z_1, z_2)\}$$

Since $0 \le q < 1$, the sequences $(x_n), (y_n)$ and (z_n) are Cauchy sequences with limit α, β and γ in X, Y and Y respectively.

Suppose that the mapping S is continuous. Then by

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} z_n$$

we get:

$$S\beta = \gamma \qquad (7)$$

By (1), for $y = \beta$ and $x = x_n$ we get:

$$[1 + pd_1(x_n, RS\beta) + pd_2(\beta, y_{n+1})]d_1(RS\beta, x_{n+1}) \le$$
$$\le p[d_1(x_n, RS\beta)d_3(S\beta, z_{n+1}) + d_1(x_n, x_{n+1})d_2(\beta, TRS\beta) + d_1(x_n, RS\beta)d_2(\beta, y_{n+1})] +$$
$$+ q \max\{d_1(x_n, RS\beta), d_1(x_n, x_{n+1}), d_2(\beta, y_{n+1}), d_3(\gamma, S\beta)\}$$

Letting n tend to infinity, by the fact that $S\beta = \gamma$ we get:

$$[1+pd_1(\alpha,RS\beta)]d_1(\alpha,RS\beta)\le qd_1(\alpha,RS\beta)$$

$$d_1(\alpha,RS\beta)\le \frac{q}{1+pd_1(\alpha,RS\beta)}d_1(\alpha,RS\beta)$$

from which it follows:

$$d_1(\alpha,RS\beta)=0 \Leftrightarrow RS\beta=\alpha \qquad (8)$$

since

$$\frac{q}{1+pd_1(\alpha,RS\beta)}\le q<1.$$

By (2), for $z=S\beta$ and $y=y_n$, we get:

$$[1 + pd_2(y_n,TRS\beta)+pd_3(S\beta,z_n)]d_2(TRS\beta,y_{n+1})\le$$
$$\le p[d_2(y_n,TRS\beta)d_1(RS\beta,x_n)+d_2(y_n,y_{n+1})d_3(S\beta,STRS\beta)+d_2(y_n,TRS\beta)d_3(S\beta,z_n)]+$$
$$+ q\max\{d_2(y_n,TRS\beta),d_2(y_n,y_{n+1}),d_3(S\beta,z_n),d_1(x_n,RS\beta)\}$$

Letting n tend to infinity, by (7) and (8) we get:

$$[1+pd_2(\beta,TRS\beta)]d_2(TRS\beta,\beta)\le qd_2(TRS\beta,\beta)$$

from which it follows $d_2(TRS\beta,\beta)=0$ or

$$TRS\beta=\beta \qquad (9)$$

By (7), (8), (9) we get:

$$TRS\beta = TR\gamma = T\alpha=\beta$$
$$STR\gamma = ST\alpha = S\beta=\gamma$$
$$RST\alpha = RS\beta = R\gamma=\alpha.$$

Thus, we proved that the points α,β,γ are fixed points of RST,TRS and STR respectively.

In the same conclusion we would arrive if one of the mappings R or T would be continuous.

We emphasize the fact that it is sufficient the contiunity of only one of the mappings T,S and R.

Let we prove now the unicity of the fixed points α,β and γ.

Assume that there is α' a fixed point of RST different from α.

By (1), for $y=T\alpha$ and $x=\alpha'$, we get:

$$[1 + pd_1(\alpha',RST\alpha)+pd_2(T\alpha,T\alpha')]d_1(RST\alpha,RST\alpha')\le$$
$$\le p[d_1(\alpha',RST\alpha)d_3(ST\alpha,ST\alpha')+d_1(\alpha',RST\alpha')d_2(T\alpha,TRST\alpha)+d_1(\alpha',RST\alpha)d_2(T\alpha,T\alpha')]+$$
$$+ q\max\{d_1(\alpha',RST\alpha),d_1(\alpha',RST\alpha'),d_2(T\alpha,T\alpha'),d_3(ST\alpha',ST\alpha)\}$$

or

$$[1 + pd_1(\alpha',\alpha) + pd_2(T\alpha,T\alpha')]d_1(\alpha,\alpha') \le$$
$$\le p[d_1(\alpha',\alpha)d_3(ST\alpha,ST\alpha') + 0 + d_1(\alpha',\alpha)d_2(T\alpha,T\alpha')] +$$
$$+ q\max\{d_1(\alpha',\alpha), 0, d_2(T\alpha,T\alpha'), d_3(ST\alpha',ST\alpha)\}$$

or

$$[1 + pd_1(\alpha,\alpha')]d_1(\alpha,\alpha') \le$$
$$\le p[d_1(\alpha,\alpha')d_3(ST\alpha,ST\alpha') +$$
$$+ q\max\{d_1(\alpha,\alpha'), d_2(T\alpha,T\alpha'), d_3(ST\alpha,ST\alpha')\} \quad (10)$$

In respect of $\max\{d_1(\alpha,\alpha'), d_2(T\alpha,T\alpha'), d_3(ST\alpha,ST\alpha')\} = \max A$ we distinguish the following three cases:

Case 1. If $\max A = d_1(\alpha,\alpha')$, we have $d_3(ST\alpha,ST\alpha') \le d_1(\alpha,\alpha')$, and by (10) we obtain:

$$[1 + pd_1(\alpha,\alpha')]d_1(\alpha,\alpha') \le$$
$$\le p[d_1(\alpha,\alpha')d_3(ST\alpha,ST\alpha') + qd_1(\alpha,\alpha') \le$$
$$\le pd_1(\alpha,\alpha')d_1(\alpha,\alpha') + qd_1(\alpha,\alpha').$$

By the above we obtain $d_1(\alpha,\alpha') \le qd_1(\alpha,\alpha')$ and since $0 \le q < 1$ we get:

$$\alpha = \alpha' \quad (11)$$

Case 2. If $\max A = d_2(T\alpha,T\alpha')$, we have $d_3(ST\alpha,ST\alpha') \le d_2(T\alpha,T\alpha')$, and by (10) we obtain:

$$[1 + pd_1(\alpha,\alpha')]d_1(\alpha,\alpha') \le$$
$$\le p[d_1(\alpha,\alpha')d_3(ST\alpha,ST\alpha') + qd_2(T\alpha,T\alpha') \le$$
$$\le pd_1(\alpha,\alpha')d_2(T\alpha,T\alpha') + qd_2(T\alpha,T\alpha').$$

or

$$[1 + pd_1(\alpha,\alpha')]d_1(\alpha,\alpha') \le [q + pd_1(\alpha,\alpha')]d_2(T\alpha,T\alpha')$$
$$d_1(\alpha,\alpha') \le \frac{q + pd_1(\alpha,\alpha')}{1 + pd_1(\alpha,\alpha')}d_2(T\alpha,T\alpha').$$

from which it follows:

$$d_1(\alpha,\alpha') \le rd_2(T\alpha,T\alpha') \quad (12)$$

where

$$0 \le r = \frac{q + pd_1(\alpha,\alpha')}{1 + pd_1(\alpha,\alpha')} < 1,$$

since $0 \le q < 1$.

Case 3. If $\max A = d_3(ST\alpha, St\alpha')$, then the inequality (10) takes the form:

$$[1+pd_1(\alpha,\alpha')]d_1(\alpha,\alpha') \le p[d_1(\alpha,\alpha')d_3(ST\alpha,ST\alpha')+qd_3(ST\alpha,ST\alpha')$$

$$d_1(\alpha,\alpha') \le \frac{q+pd_1(\alpha,\alpha')}{1+pd_1(\alpha,\alpha')}d_3(ST\alpha,ST\alpha')$$

$$d_1(\alpha,\alpha') \le rd_3(ST\alpha,ST\alpha') \qquad (13)$$

Continuing our argumentation for the Case 2, by (2) for $z=ST\alpha$ and $y=T\alpha'$ we have:

$$\begin{aligned}&[1+pd_2(T\alpha',TRST\alpha)+pd_3(ST\alpha,ST\alpha')]d_2(TRST\alpha,TRST\alpha')\le\\ &\le p[d_2(T\alpha',TRST\alpha)d_1(RST\alpha,RST\alpha')+d_2(T\alpha',TRST\alpha')d_3(ST\alpha,STRST\alpha)+\\ &\quad+d_2(T\alpha',TRST\alpha)d_3(ST\alpha,ST\alpha')]+q\max\{d_2(T\alpha',TRST\alpha),\\ &\quad d_2(T\alpha',TRST\alpha'),d_3(ST\alpha,ST\alpha'),d_1(RST\alpha',RST\alpha)\}\end{aligned}$$

or

$$\begin{aligned}&[1+pd_2(T\alpha',T\alpha)+pd_3(ST\alpha,ST\alpha')]d_2(T\alpha,T\alpha')\le\\ &\le p[d_2(T\alpha',T\alpha)d_1(\alpha,\alpha')+d_2(T\alpha',T\alpha')d_3(ST\alpha,ST\alpha)+\\ &\quad+d_2(T\alpha',T\alpha)d_3(ST\alpha,ST\alpha')]+q\max\{d_2(T\alpha',T\alpha),\\ &\quad d_2(T\alpha',T\alpha'),d_3(ST\alpha,ST\alpha'),d_1(\alpha',\alpha)\}\end{aligned}$$

or

$$\begin{aligned}&[1+pd_2(T\alpha',T\alpha)]d_2(T\alpha,T\alpha')]\le pd_2(T\alpha',T\alpha)d_1(\alpha,\alpha')+\\ &+q\max\{d_1(\alpha',\alpha),d_2(T\alpha',T\alpha),d_3(ST\alpha,ST\alpha')\}\end{aligned}$$

or

$$[1+pd_2(T\alpha',T\alpha)]d_2(T\alpha,T\alpha')\le pd_2(T\alpha',T\alpha)d_1(\alpha,\alpha')+q\max A \qquad (14)$$

In the Case 2, we have $\max A=d_2(T\alpha,T\alpha')$ and by (14) we obtain:

$$[1+pd_2(T\alpha',T\alpha)]d_2(T\alpha,T\alpha')\le pd_2(T\alpha',T\alpha)d_2(T\alpha',T\alpha)+qd_2(T\alpha,T\alpha')$$

or

$$d_2(T\alpha,T\alpha')\le qd_2(T\alpha,T\alpha').$$

Since $0\le q<1$, we obtain:

$$d_2(T\alpha,T\alpha')=0$$

and by (13) it follows that $d_1(\alpha,\alpha')=0$, so we obtain again the inequality (11).

In the Case 3, by (3) for $x=RST\alpha, z=ST\alpha'$ and in the same way we obtain:

$$[1+pd_3(ST\alpha',ST\alpha)]d_3(ST\alpha,ST\alpha')\le pd_3(ST\alpha',ST\alpha)d_2(T\alpha',T\alpha)+q\max A.$$

Since in this case $\max A=d_3(ST\alpha,ST\alpha')$, we have $d_2(T\alpha',T\alpha)\le d_3(ST\alpha,ST\alpha')$ and we obtain:

$$[1+pd_3(ST\alpha,ST\alpha')]d_3(ST\alpha,ST\alpha')\le pd_3(ST\alpha,ST\alpha')d_3(ST\alpha,ST\alpha')+qd_3(ST\alpha,ST\alpha')$$

from which it follows

$$d_3(ST\alpha,ST\alpha')\le qd_3(ST\alpha,ST\alpha').$$

Since $0 \le q < 1$ we take:

$$d_3(ST\alpha, ST\alpha') = 0$$

and by (13) it follows $d_1(\alpha, \alpha') = 0$. Thus, again, in this case the following equality holds:

$$\alpha = \alpha' .$$

In the same way, it is proved the unicity of β and γ.

Application. Let $X = Y = Z = [0,1] \subset R$ and the mappings defined as follows:

$$Tx = x, Rz = 1 \text{ and } Sy = \begin{cases} 1 & \text{for } y \in]0,1] \\ \frac{1}{2} & \text{for } y = 0 \end{cases}$$

We have:

$$RSy = 1, TRz = 1 \text{ and } STx = \begin{cases} 1 & \text{for } x \in]0,1] \\ \frac{1}{2} & \text{for } x = 0 \end{cases}$$

$$RSTx = 1, TRSy = 1 \text{ and } STRZ = 1.$$

We have to show now that T, R and S satisfy the conditions of Theorem 2.1 for $k = \frac{1}{2}$ and $q = \frac{3}{4}$. Indeed:

For every $x, y \in [0,1]$ we have $d_1(RSy, RSTx) = |1-1| = 0$. Then the verity of the inequality (1) is clear since its left side is 0.

The verity of the inequality (2) is clear too, since $d_2(TRz, TRSy) = 0, \forall x, y \in [0,1]$.

We consider now the inequality (3). We have

$$d_3(STx, STRz) = \begin{cases} |\frac{1}{2} - 1| = \frac{1}{2}, & \text{for } x = 0 \text{ and } 0 \le z \le 1 \\ |1-1| = 0, & \text{for } 0 < x \le 1 \text{ and } 0 \le z \le 1 \end{cases}$$

We distinguish two cases:

Case 1. For $x = 0$ and $0 \le z \le 1$, the inequality (3) takes the form

$$(1 + p|z - \frac{1}{2}| + p|0-1|)\frac{1}{2} \le p(|z - \frac{1}{2}| - |0-1| + |z-1| \cdot |0-1| + |z - \frac{1}{2}| \cdot |0-1|) +$$

$$+ q \max\{|z - \frac{1}{2}|, |z-1|, |0-1|, |1-0|\}.$$

We get

$$(1+p)\frac{1}{2} \le \frac{3p}{2}|z - \frac{1}{2}| + p|z-1| + q .$$

For $p=\frac{1}{2}$ and $q=\frac{3}{4}$, we obtain

$$\frac{3}{4}\le\frac{3}{4}|z-\frac{1}{2}|+\frac{1}{2}|z-1|+\frac{3}{4}$$

or

$$0\le\frac{3}{4}|z-\frac{1}{2}|+\frac{1}{2}|z-1|$$

for all $z\in[0,1]$.

Thus, the inequality (3) is satisfied.

Case 2. For $0<x\le 1$ and $0\le z\le 1$, since $d_3(STx,STRz)=0$, the inequality (3) is satisfied.

Therefore, as a conlcusion, we have the mappings T,S and R satisfy all the conditions of the Theorem 2.1 for $p=\frac{1}{2}$ and $q=\frac{3}{4}$. The unique fixed point is 1 for each of the mappings RST,TRS and STR.

Corollary 2.2 *Theorem 1.2*[2] *is taken by Theorem 2.1 for* $p=0$. *Further, it is sufficient the continuity of only one of the three mappings.*

References

[1] S. Č. Nešic', *Common fixed point theorems in metric spaces*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), **46(94)** (2003), No.3-4, 149-155.

[2] N. P. Nung, *A fixed-point theorem in three metric spaces*, Math. Sem. Notes, Kobe Univ. **11**(1983), 77-79.

[3] R. K. Jain, A. K. Shrivastava and B. Fischer, *Fixed points on three complete metric spaces*, Novi Sad J. Math. **27**(1977), No. 1, 27-35.

Optimal (r, nQ, T) Inventory Control under Stationary Demand

A.G. Lagodimos
University of Piraeus, Department of Business Administration, 80 Karaoli and Dimitriou Street, GR-185 34 Piraeus, Greece
Email: alagod@unipi.gr

I. T. Christou
Athens Information Technology, Markopoulou Avenue (19th Km), GR-190 02 Paiania, Greece
Email: ichr@ait.edu.gr

K. Skouri
University of Ioannina, Department of Mathematics, GR-45 110, Ioannina Greece
Email: kskouri@uoi.gr

Abstract. We consider the control of a single-echelon inventory system under the (r, nQ, T) ordering policy. Demand follows a stationary stochastic process and, when unsatisfied, is backordered. Under a standard cost structure, our aim is the minimization of the total average cost. In contrast to previous research, all policy variables (i.e. reorder level r, batch size Q and review interval T) are simultaneously optimized. While total average cost is not convex, two new convex bounds together with a Newsboy characterization of the optimal solution lead to an exact algorithm with guaranteed convergence to the global optimum. Computational results demonstrate that the inclusion of the review interval as a decision variable in the optimization problem can offer serious cost savings.

Keywords: Supply chain; Newsboy; Inventory; Optimal; Stochastic demand.

1. Introduction

The practical value of periodic review inventory control policies, where ordering decisions are taken in regular time intervals, is well established (e.g. Silver et al. 1998). Although theoretically inferior from their continuous review counterparts in terms of average cost performance (e.g. Veinott 1966 and Lee and Nahmias 1993), periodic policies offer practical advantages. By not imposing continuous monitoring of inventory status, they can be easily implemented in real production environments. By allowing for the routine overshoot of the reorder point, respective models can accommodate lumpy demands without loss of modeling accuracy. Hence, it is not surprising that the MRP logic, designed to deal with such demand processes, effectively implements standard periodic review ordering policies (e.g. Anderson and Lagodimos 1989 and Axsater and Rosling 1994).

Focusing on an established periodic review policy, the (r,nQ,T) policy, in this paper we study the optimal policy variables determination for a single-echelon inventory installation and propose an algorithm that guarantees cost-optimal control. While the system operating assumptions (i.e. stationary demand, full backordering and cost structure) are common in the field, there is an important differentiation from previous research. Earlier work has considered the (r, nQ) policy, a reduced version of the (r,nQ,T) policy, which assumes a fixed review interval T and where only two policy variables need be determined (the reorder level r and the batch size Q). In this paper all three policy variables (r,Q,T) of the original policy are simultaneously optimized, leading to solutions offering serious cost savings over those of the reduced problem.

We start with a review of key previous findings. The (r,nQ,T) policy was originally proposed by Morse (1959) as an adaptation of the (R,T) policy (also known as base-stock) to cope with quantized orders. These occur when supplies are constrained to be multiples of some basic batch-size Q, usually reflecting some physical limitation of the supply process (e.g. pallet-load, container load etc.). Considering the form of the policy, Veinott (1965) demonstrated that the (s,S) policy is optimal for externally fixed batch-size and zero ordering cost. Otherwise, the policy is clearly inferior compared to the more general policy, which is the known optimal policy for unrestricted supplies (e.g. Veinott 1966 and Federgruen and Zipkin 1985). This was further supported by the numerical results in Wagner et al. (1965) and Veinott and Wagner (1965) but who also suggested respective cost difference not to be large when comparing both policies at their optimal setting.

For independent demands, Hadley and Whitin (1961), building on the results by Morse, used Markov steady-state analysis to characterize the distribution of the inventory position of the (r,nQ,T) policy as being uniform $U(r, r+Q)$ irrespective of the demand distribution. As recently proposed by Li and Sridharan (2008), this distribution remains unchanged even for serially correlated demands. Using a general cost structure (the one also used here), Hadley and Whitin (1963) modelled long-run average cost of the (r,nQ,T) policy for Normal and Poisson demands. They also studied the cost function in terms of the policy variables (r, Q,T). No analytical properties for the total cost were determined, so concluded that optimal control can only rely on exhaustive search approaches. Numerical comparisons with the (r,nQ,T) policy showed only marginal cost-differences as well as a tendency for the policy to often degenerate to (R, T) at its optimal setting.

Considering the (r, nQ) policy, Zheng and Chen (1992) proposed an algorithm to compute the ordering parameters (r and Q) that minimize long-run average cost for discrete uncorrelated demands. Under the cost structure in Hadley and Whitin but omitting review costs (since the policy assumes a fixed T), they proved average cost convexity in the reorder-level r together with a Newsboy-styled condition at the optimum. While the cost behaviour in Q was found erratic, Zheng and Chen proposed a convex bound, effectively identical to the average cost function in continuous review (r, Q) policies (e.g. Zipkin 2000). Implementing the algorithm, they numerically compared the (r, nQ) and (s,S) policies and only found small cost differences at the respective optima.

These papers, as well as a more recent by Larsen and Kiesmuller (2007) that models average cost for the (r, nQ) policy under Erlang demand, all considered the review interval T as fixed, thus not entering the optimal control problem. The only previous study that (to our knowledge) considers T as a control variable is that by Rao (2003). Focusing on the (R, T) policy under uncorrelated stationary demands, he showed that this can be analyzed as a limiting case of the continuous review (r, Q) policy. This allowed him to show that the total average cost for the (R, T) policy is jointly convex in both the order-up-to level R and the review interval T. Therefore any convex optimization algorithm can be used for optimal system control.

In this paper considering the (r, nQ, T) policy in its unrestricted form we propose an algorithm that guarantees convergence to the global optimum in all three policy variables.

The rest of the paper is organised as follows: In section 2 we present the notation and assumptions used through the paper. In section 3 the (r, nQ, T) policy's total average cost is modelled. In section 4 bounds for the total average cost are proposed and some properties of these bounds are obtained. An algorithm for the determination of the optimal policy's parameters is presented in section 5. In section 6 we give examples for the determination of the optimal controls assuming Normal distributed demand, under different cost settings. Finally some conclusions are given in section 7.

2. Notation and Assumptions

In this section we introduce the notation used together with the assumptions underlying the operation of the inventory system.

2.1 Notation

μ	The demand rate.
$D(t)$	Random variable denoting cumulative demand through time t, i.e demand in the interval [0, t] .
h, p	Inventory holding and backordering cost per unit per unit time respectively.
Ko	Fixed ordering cost (per ordering decision).
Kr	Fixed review cost (per review).
L	Replenishment lead time.
R	Upper starting inventory position limit (just after a review).
Q	Basic batch size (just after a review).
$I(R, Q, t)$	Net inventory position at time t.
P_o	Probability of ordering at any review period.
T	Length of review interval.
α	The α-service measure (non-stock-out probability).
x^+, x^-	The functions $x^+ = \max\{0, x\}$ $x^- = \max\{0 - x\}$.
$C(R, Q, T)$	Total average cost (per review interval).

2.2 Operating Assumptions

We study a single-item, single-echelon inventory installation controlled by *(r,nQ,T)* policy. Inventory status is reviewed every a time interval of length *T*. In the event an order is placed after a review, the order quantity is available after the elapse of a, deterministic and known, replenishment lead time *L*. Material leaves the installation in response to specific customer demands. Demand not satisfied from stock is backordered. The demand process is stochastically non-decreasing in t with mean $t\mu$, density $f(x,t)$ and cumulative distribution function $F(x, t)$. As is common in modelling cost we assume that cumulative demand has stationary and independent increments (see Serfozo and Stidham (1978)). These assumptions hold if the demand is modelled, for example, either as compound Poisson or Normal processes (see Rao (2003)). Note that the same assumptions for demand process are also used in Zipkin (1986) to obtain convexity properties for *(r, Q)* policy.

We also need to clarify the sequence of events within any review interval. (1) Replenishment orders placed respective lead time L earlier are received. (2) Inventory status is reviewed and a replenishment decision is taken. (3) Demand is realized. (4) Inventories and backorders are measured and relevant costs evaluated.

3. The *(r, nQ, T)* Inventory Policy

The *(r,nQ,T)* policy operates as follow: a) Inventory status is reviewed and ordering decisions taken at regular intervals of length *T*; b) If the inventory position is found to be below a reorder level r, a replenishment order is placed; c) Irrespectively of the taken decision the starting inventory position after any review epoch is given by $R\text{-}X(Q)$, and follows a uniform distribution U(r, R), where $X(Q)$ follows a uniform distribution $U(0,Q)$ and $R=r+Q$.

The size of an order (if it is placed) is nQ, where Q a predetermined batch size and n the smallest integer for which $R\text{-}X(Q) \geq r$. Note that for independent demands, it has been established (see Hadley and Whitin 1963) that $R\text{-}X(Q)$ follows a uniform distribution U(r, r+Q). As it was recently shown, the same distribution holds even for time-correlated demands (Li and Sridharan, 2008).

Remark 1. It is worthwhile to note that an alternative definition for *(R, T)* policy can be deduced setting Q= 0.

This remark allow a unified treatment for *(r, nQ, T)* and *(R, T)* policies.

3.1. The cost function under *(r,nQ,T)* policy

In this sub-section we model the average cost of the system and we obtain the expression for α-measure, which leads later to the Newsboy characterization of the optimal policy. We consider the general four element cost structure proposed in the seminal analysis by Hadley and Whitin (1963). In this context we assume linear holding and backordering penalty costs, as well as two fixed cost elements: ordering cost per actual replenishment order and review cost per review occasion. The review cost, K_r, incurred every T time units at each review and the ordering cost, K_o, incurred at the review instants where actual replenishment orders are released (so respective cost coefficient is multiplied by the ordering probability P_o). As discussed in Hadley and Whitin (1963), P_o represents the probability that demand between two consecutive reviews triggers an order at the second review so this clearly implies that $P_o=Pr(Q - X(Q) < D(T))$. Observe that P_o depends on Q and T but does not depend on R.

So, the inventory holding and backordering costs at time $t \in [0,T]$, have been pooled into the following function:

$$G(R,Q,t) = hE[(I(R,Q,t)^{+}] + pE[(I(R,Q,t)^{-}]$$

$$= hE[R - X(Q) - D(L+t))^{+}] + E[R - X(Q) - D(L+t))^{-}]$$

$$= h(R - \frac{Q}{2} - \mu(L+t)) + (h+p)E[(X(Q) + D(L+t) - R)^{+}]$$

$$= h(R - \frac{Q}{2} - \mu(L+t)) + \frac{h+p}{Q}\int_{R}^{+\infty}(y-R)\int_{0}^{Q} f(y-x,t)dxdy \quad (1)$$

note that we use the facts that

$$E[(I(R,\ Q,\ t)] = E[(I(R,\ Q,\ t)^{+}] + E[(I(R,\ Q,\ t)^{-}]$$

and

$X(Q) + D(L+t)$ has density

$$\frac{1}{Q}\int_{0}^{Q} f(y-x,t)dx$$

Thus the average holding and backordering costs per unit time is

$$H(R, Q,T) = \frac{1}{T}\int_0^T G(R,Q,t)dt \tag{2}$$

and consequently the average total system cost per unit time can be expressed as:

$$C(R,Q,T) = \frac{K_r + K_o P_o}{T} + H(R,Q,T) \tag{3}$$

Now we obtain the expression for α-measure of service. Since the α-measure is defined as the non stock out probability we can easily see that

$$a(R,Q,T) = 1 - \frac{1}{T}\int_0^T \Pr(R - X(Q) - D(L+t)) \leq 0dt \tag{4}$$

4. Total Cost's Bounds and Properties

In this section we introduce two bounds to the cost function and present analytic properties that form the basis for the system optimal control.

Since the ordering probability always satisfies the relation, $0 \leq P_o \leq 1$, the following two bounds for $C(R,Q,T)$ directly prevail:

$$B_L(R,Q,T) = \frac{K_r}{T} + H(R,Q,T) \tag{5}$$

and

$$B_U(R,Q,T) = \frac{K_r + K_o}{T} + H(R,Q,T) \tag{6}$$

In the next lemma the optimal value of R is determined for given values of T and Q.

Lemma 1. Let $R(Q,T) = \arg\min_R C(R,Q,T)$ denotes the optimal R corresponding to fixed T and Q, then *R(Q, T)* satisfies the condition $a(R(Q,T),Q,T) = \frac{p}{h+p}$, where $a(R(Q,T),Q,T)$ is the α-measure for this T and Q values.

Proof. From $\frac{dC(R,Q,T)}{dR} = 0$ we obtain the following Newsboy style equation

$$h - \frac{h+p}{T}\int_0^T \Pr(R - X(Q) - D(L+t)) \leq 0dt = 0$$

or

$$h - (h+p)(1 - a(R,Q,T)) = 0$$

and finally we get

$$a(R(Q,T),Q,T) = \frac{p}{h+p} \tag{7}$$

Direct application of this result clearly reduces the problem state-space (by one variable), thus facilitating optimal control parameters.

Notice that the optimal R for fixed T and Q is the same for the average cost per unit time, $C(R, Q, T)$, and for its bounds, $B_L(R, Q, T)$ and $B_U(R, Q, T)$ so the next lemma follows easily:

Lemma 2. $R(Q,T) = \arg\min_R C(R,Q,T) = \arg\min_R B_L(R,Q,T) = \arg\min_R B_U(R,Q,T)$

Lemma 3. For every given T let $R(Q;T) = \arg\min_R C(R, Q,T)$ denotes the optimal R corresponding to Q, then $B_L(R(Q;T), Q, T)$ and $B_U(R(Q;T), Q, T)$ are increasing and convex functions in Q.

Proof. Zheng (1992) in Lemma 4 prove that the function $G(R(Q;T), Q, t)$ is an increasing and convex function of Q so the same holds for the functions $B_L(R(Q;T), Q, T)$ and $B_U(R(Q;T), Q, T)$ (see also figure 1).

It is worthwhile to note that $B_L(R, Q, T)$ and $B_U(R, Q, T)$ represent the cost for systems, which operate under a (r, nQ, T) policy but it forces to order at each review epoch (under different fixed costs). In such circumstance it is know that these systems require $Q=0$ (Veinott 1966) and consequently it is optimal for these systems to operate under (R, T) policies.

Lemma 4. Let $R(0,T) = \arg\min_R C(R, 0,T)$ denotes the optimal R corresponding to T for Q=0, if $D(t)$ is stochastically increasing linearly then $B_L(R(0,T), 0, T)$ and $B_U(R(0,T), 0, T)$ are convex function in T.

Proof. From Rao (2003) Theorem 6, $B_L(R, 0, T)$ and $B_U(R, 0, T)$ are jointly convex in R and T so $B_L(R(0,T), 0, T)$ and $B_U(R(0,T), 0, T)$ are convex function in T (see also figure 2).

It is interesting to observe that in the above both $B_L(R(0,T), 0, T)$ and $B_U(R(0,T), 0, T)$ represent the optimal cost of (R,T) policies (by definition obtained from the (r,nQ,T) policy with $Q = 0$) for given T. Therefore, the optimal (r,nQ,T) policy is bounded above and below by specific (R,T) policies derived from the bounds in (5) and (6).

The next result follows naturally.

Lemma 5. $\lim_{T\to\infty}[B_U(R, Q, T) - B_L(R, Q, T)] = 0$

5. Optimal Control

In this section we present an algorithm for the determination of the optimal values for (*R, Q, T*). This algorithm converges to the optimal in a finite number of steps for any given accuracy.

`Algorithm` $R^*, Q^*, T^* = Min < R, nQ, T > Cost(K_r, K_o, L, \mu, \sigma, h, p)$

`inputs:` review cost, ordering cost, lead-time, demand distribution mean and variance, holding and backorders cost coefficients

`outputs:` control parameters for order-up-to level R, order quantum Q (in multiples of a quantum order Q_q, review period interval T (in multiples of a time quantum T_q)

```
1. set C* = +∞
2. for T=T_q, 2T_q... do
     a. for Q=Q_q, 2Q_q... do
          i. let R' = argmin_R C(R,Q,T);
         ii. let C^{Q,T} = C(R',Q,T), B_L^{Q,T} = min_R B_L(R,Q,T);
        iii. if C^{Q,T} < C*
                1. set C* = C^{Q,T}, R* = R', Q* = Q, T* = T;
         iv. end if;
          v. if B_L^{Q,T} > C* break;
     b. end for;
     c. let B_T* = min_{R,Q} B_L(R,Q,T);
     d. if B_T* > C* break;
3. end for
4. return (R*,Q*,T*);
End.
```

The algorithm is guaranteed to terminate as the lower bound of the function goes to infinity as $T \to +\infty$ and *$B_L(R(Q;T), Q, T)$* is increasing and convex functions in *Q* and also

$\lim_{Q\to\infty} B_L(R_{Q,T},Q,T) = +\infty$. Therefore, the conditions in steps 2.a.v as well as 2.d will eventually be met and the algorithm will terminate. The conditions are also sufficient:

1. For the case of step 2.a.v, there is no point in searching for any higher Q as it is guaranteed that the cost function, being greater than the lower bound will always be greater than our current incumbent value, as all other values in the range $[T,\ldots]$x$[Q,\ldots]$ will yield higher costs (the lower bound is now increasing in Q).

2. For the case of step 2.d it is obvious that at the value of T for which the condition is met, the sequence B_T^* is increasing (otherwise it would have been impossible to have found a cost value less than the lower bound) and thus, from now on the sequence $C_T^* = \min_{(R,Q)} C(R,Q,T)$ will always be above the current C^* which becomes the global optimum.

The previous results summarized in the following:

Proposition. The proposed algorithm converges to the optimal in a finite number of steps.

6. Numerical Results

We have applied the proposed procedure to determine the optimal controls assuming Normal distributed demand with $E(D(t)) = t\mu$ and $Var(D(t)) = t\sigma^2$, with μ=10, σ=3, L=5 for a number of different cost coefficients (the formula for total average cost assuming Normal distributed demand is derived in the Appendix). In addition note (see Rao, 2003) that all feasible reorder intervals must be at least $t_{\min}$ and demand rate μ is sufficiently larger than σ ($\mu > 3\sigma$) so that $\mu t >> \sigma\sqrt{t}$ for al $t \geq t_{\min}$ and consequently the probability of negative demand is negligibly small for $T \geq t_{\min}$. The results are shown in Table 1 below. The first four columns in Table 1 determine the cost coefficients of the problem. The columns entitled Ropt and Topt under the heading "(R,T) Policy Optimization" are the optimal controls of the (R,T) policy applied to the problem, and RTcost is the optimal cost of the (R,T) policy. The columns r^*, Q^* and T^* under the heading "(R,nQ,T) Policy Optimization" denote the optimal controls for the policy (R,nQ,T) when all three parameters are allowed to vary, and the column denoted $C^*(R^*,Q^*,T^*)$ denotes the optimal value of the cost function of the (R,nQ,T)

policy. Finally, the last 3 columns determine the optimal controls R^* and Q^* together with the value for the optimal (R,nQ,T) policy when the time parameter (the length of the period) is arbitrarily set to 1 and not allowed to vary.

The rows in bold are the cases where the (R,nQ,T) policy is strictly better than the simpler (R,T) policy. As can be seen, the differences of the policies in terms of the optimal cost are relatively small in all cases; in most cases, the optimal (R,nQ,T) policy reduces to the (R,T) policy. Nevertheless, notice the important role the T parameter (length of period) can play in the optimal cost determination. For example, for the case K_r=250, K_o=1, h=10, p=1, the optimal (R,nQ,T) policy is more than 270% better than the optimal policy determined by fixing the parameter T=1 !.

7. Conclusions

In this paper we developed an algorithm for computing optimal *(r, nQ, T)*. This algorithm is constructed incorporating results for *(r, Q)* and *(R, T)* policies. To the best of our knowledge only results for *(r, nQ)* policy, a special case of *(r, nQ, T)* policy with T=1, have been presented by Zheng and Chen (1992) and recently by Larsen Kiesmuller (2007). The computational findings presented in the previous section indicate serious cost savings when the parameter T is a decision variable. In addition a close relation between the *(r, nQ, T)* and *(R, T)* is concluded. From $B_U(R,Q,T)$ the optimal cost of the *(R, T)* is an upper bound for the optimal cost of *(r, nQ, T)*. While the numerical results show that in many cases the two costs coincide.

Table 1. Optimal Control of *(R,T)*, *(R,nQ,T)* & *(R,nQ,T*=1) Policies

L=5, μ=10, σ=3				(R,T) Policy Optimization			(r,nQ,T) Policy Optimization				(r,nQ,T=1) Policy		
Kr	Ko	h	p	Ropt	Topt	RTcost	r*=R*-Q*	Q*	T*	C*(R*,Q*,T*)	r*=R1*-Q1*	Q1*	C*(R*,Q*,T=1)
1	1	1	1	56,1	1,24	7,98	55,9	0	1,2	7,98	54,93	0	8,07
1	1	1	10	64,43	0,89	15,79	64,53	0	0,9	15,79	65,22	0	15,83
1	1	1	100	70,93	0,76	22,32	71,29	0	0,8	22,33	72,95	0	22,59
1	1	10	1	44,79	0,97	15,51	44,89	0	1	15,51	44,89	0	15,51
1	1	10	10	52,57	0,52	60,04	52,48	0	0,5	60,02	54,93	0	62,73
1	1	100	1	37,3	0,87	21,78	37,36	0	0,9	21,78	37,61	0	21,83
1	50	1	1	73,12	4,71	24,01	73,07	0	4,7	24,01	31,5	47	24,62
1	**50**	**1**	**10**	**85,28**	**3,51**	**37,62**	**53,17**	**37**	**0,8**	**37,26**	54,36	37	37,34
1	**50**	**1**	**100**	**93,3**	**3,17**	**46,96**	**62,09**	**35**	**0,6**	**45,56**	65,02	35	46,02
1	50	10	1	50,78	3,67	35,96	50,82	0	3,7	35,96	18,63	37	37,22
1	**50**	**10**	**10**	**58,71**	**1,78**	**100,21**	**43,49**	**18**	**0,5**	**100,07**	45,46	19	102,69
1	**50**	**10**	**100**	**67,26**	**1,28**	**186,37**	**55,18**	**14**	**0,3**	**182,28**	65,22	0	189,33
1	50	100	1	41,19	3,49	43,69	41,21	0	3,5	43,69	10,28	35	45,54
1	50	100	10	46,08	1,39	181,22	46,09	0	1,4	181,22	44,89	0	186,11
1	250	1	1	100,1	10,11	51	100,05	0	10,1	51	4,5	101	51,58
1	**250**	**1**	**10**	**120,49**	**7,54**	**73,77**	**48,35**	**77**	**0,9**	**72,2**	48,89	77	72,2
1	**250**	**1**	**100**	**130,55**	**6,94**	**86,54**	**60,01**	**74**	**0,7**	**82,15**	62,1	74	82,48
1	250	10	1	55,93	7,7	71,03	55,93	0	7,7	71,03	0	61	74,3
1	**250**	**10**	**10**	**66,75**	**3,42**	**176,45**	**35**	**35**	**0,5**	**175,08**	37,49	35	176,38
1	250	100	1	43,8	7,41	80,3	43,8	0	7,4	80,3	0,01	48	89,25
1	250	100	10	49,06	2,69	278,28	49,08	0	2,7	278,29	26,76	27	285,24
250	1	1	1	100,1	10,11	51	100,05	0	10,1	51	54,93	0	257,07
250	1	1	10	120,49	7,54	73,77	120,11	0	7,5	73,77	65,22	0	264,83
250	1	1	100	130,55	6,94	86,54	130,15	0	6,9	86,54	72,95	0	271,59
250	1	10	1	55,93	7,7	71,03	55,93	0	7,7	71,03	44,89	0	264,51
250	1	10	10	66,75	3,42	176,45	66,62	0	3,4	176,45	54,93	0	311,73
250	1	10	100	77,1	2,53	290,51	76,82	0	2,5	290,53	65,22	0	389,35
250	1	100	1	43,8	7,41	80,3	43,8	0	7,4	80,3	37,61	0	270,83
250	1	100	10	49,06	2,69	278,28	49,08	0	2,7	278,29	44,89	0	386,13
250	1	100	100	56,64	1,35	837,18	56,87	0	1,4	837,48	54,93	0	858,33
250	50	1	1	104,36	10,99	55,63	104,55	0	11	55,63	31,5	47	273,62
250	50	1	10	126,59	8,23	79,98	126,32	0	8,2	79,98	54,36	37	286,34
250	50	1	100	136,98	7,59	93,29	137,12	0	7,6	93,29	65,02	35	295,02
250	50	10	1	56,67	8,38	77,13	56,69	0	8,4	77,13	18,63	37	286,22
250	50	10	10	68,12	3,7	190,2	68,1	0	3,7	190,2	45,46	19	351,69
250	50	10	100	78,86	2,75	309,07	78,47	0	2,7	309,1	65,22	0	438,33
250	50	100	1	44,13	8,08	86,63	44,14	0	8,1	86,63	10,28	35	294,54
250	50	100	10	49,47	2,9	295,81	49,46	0	2,9	295,81	44,89	0	435,11
250	50	100	100	57,11	1,45	872,17	56,87	0	1,4	872,47	54,93	0	907,3
250	250	1	1	120,58	14,21	71,48	120,55	0	14,2	71,48	4,5	101	300,58
250	250	1	10	147,56	10,58	101,25	147,73	0	10,6	101,25	48,89	77	321,2
250	250	1	100	159,04	9,8	116,3	159,05	0	9,8	116,3	62,1	74	331,48
250	250	10	1	59,05	10,71	98,08	59,04	0	10,7	98,08	0	61	323,3
250	250	10	10	72,9	4,67	237,97	73,07	0	4,7	237,98	37,49	35	425,38
250	250	10	100	85,01	3,48	373,3	85,19	0	3,5	373,31	56,25	27	535,87
250	250	100	1	45,1	10,34	108,34	45,08	0	10,3	108,34	0,01	48	338,25
250	250	100	10	50,72	3,64	356,88	50,66	0	3,6	356,9	26,76	27	534,24
250	250	100	100	58,64	1,76	996,49	58,8[illegible]	0	1,8	996,64	54,93	0	1107,22

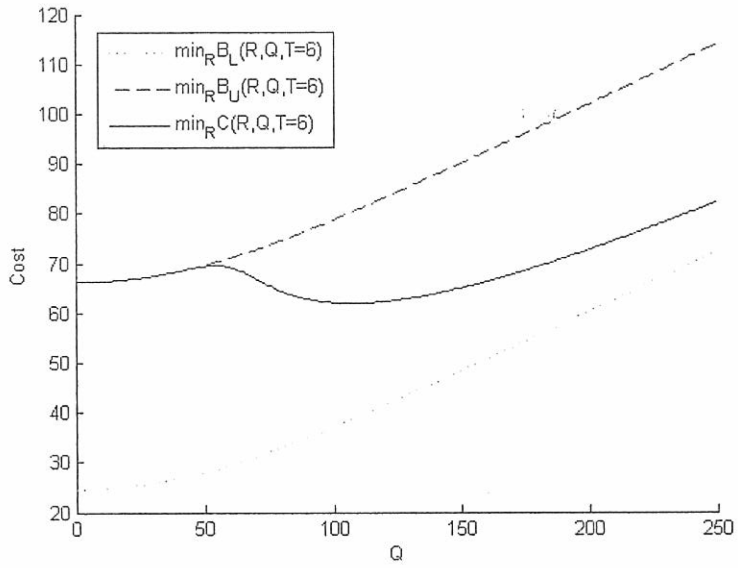

Figure 1: Plot of the cost function and its bounds as a function of the base order quantity Q at their minimum over R, for K_r=50, K_o=250, L=5,μ=10,σ=3,h=1,p=1,T=6. Both bounds are convex increasing. Notice however that the function $C(Q)$ has two local minimums (the first at Q=0). The upper bound coincides with the cost function for Q<50.

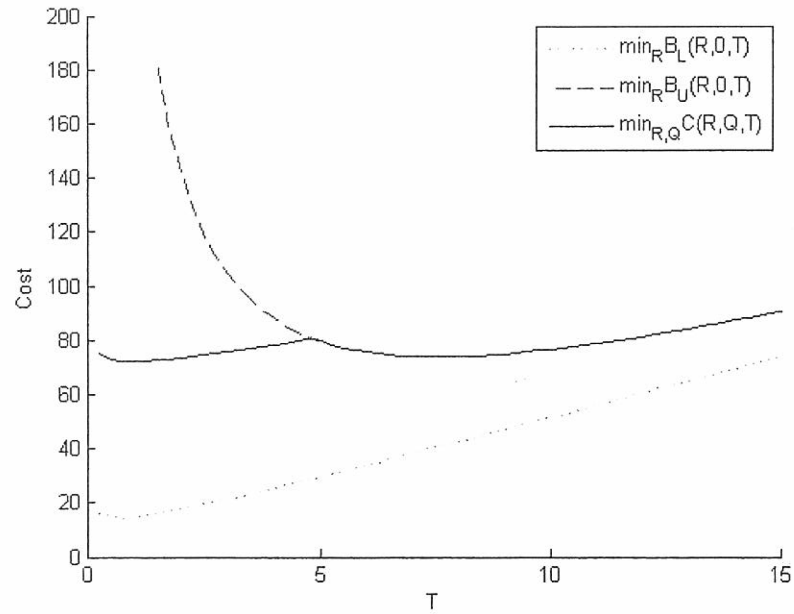

Figure 2: Plot of the cost function and its bounds as a function of the period length T at their minimum over R and Q, for K_r=1, K_o=250, L=5, μ=10, σ=3, h=1, p=10. Both bounds are convex but not increasing in T. The function $C(T)$ has again two local minimums (the first is the global minimum). Also notice that the upper bound coincides with the actual cost function for T>5.

Appendix: Closed-form expressions for Normal demand

Assuming Normal distributed demand, we now obtain the total average cost and a-service measure. Since we assume uncorrelated Normal distributed demand, the demand over t consecutive periods is also Normal with $E(D(t)) = t\mu$ and $Var(D(t)) = t\sigma^2$. In the following, we make use of the standardizing ratios:

$$Z_t = \frac{R-Q-t\mu}{\sigma\sqrt{t}}\left[=\frac{r-t\mu}{\sigma\sqrt{t}}\right],\ R_t = \frac{Q}{\sigma\sqrt{t}},\ M_t = \frac{t\mu}{\sigma\sqrt{t}} \tag{8}$$

We start by modeling average total cost *C(R, Q, T)* in (3). So firstly, we need to model the ordering probability P_o. By standardizing the variable *D(T)* this can be expressed as:

$$P_o = \Pr(u > w) = 1 - \Pr(u \le w) = 1 - \frac{1}{R_T}\int_{-M_T}^{R_T - M_T} \Phi(x)dx$$

where

$u \sim N(0,1)$ and $w \sim U(-M_T, R_T - M_T)$ and $\Phi(.)$ is the cumulative distribution function for the standard Normal.

We can directly evaluate the integral above (using integration by parts) and obtain the average fixed costs, say Θ(Q,T):

$$\Theta(Q,T) = \frac{K_r + K_o}{T} - \frac{K_o}{TR_T}(\varphi(R_T - M_T) + (R_T - M_T)\Phi(R_T - M_T) - \varphi(M_T) + M_T\Phi(-M_T) \tag{9}$$

where $\varphi(.)$ is the density function for the standard Normal. We consider now average holding and backordering cost, $H(R,\ Q,\ T)$. Since $E(R - X(Q)) = R - \frac{Q}{2} = r + \frac{Q}{2}$ we only need to determine $E(I(R,Q,t)^-)$. By standardizing the Normal variable *D(L+t)*, this can be expressed in terms of the variables $u \sim N(0,1)$ and $w \sim U(0, R_{L+t})$

So

$$E(I(I,R,t)^-) = \frac{\sigma\sqrt{L+t}}{R_{L+t}}\int_{Z_{L+t}}^{Z_{L+t}+R_{L+t}}\int_x^{\infty}(y-x)\phi(y)dydx = \frac{\sigma\sqrt{L+t}}{R_{L+\tau}}\int_{Z_{L+t}}^{Z_{L+t}+R_{L+t}}[\phi(x) - x + x\Phi(x)]dx. \tag{10}$$

Using again integration by parts (twice), after some algebra a closed-form expression for $E(I(R,Q,t)^-)$ is obtained. So, we finally get $H(R,\ Q,\ T)$ as:

$$H(R,\,Q,\,T)=h[R-\frac{Q}{2}-\mu(L+\frac{T+1}{2})]+\frac{h+p}{2T}\int_0^T\frac{\sigma\sqrt{L+t}}{R_{L+t}}\{[(Z_{L+t}+R_{L+t})^2+1]\Phi(Z_{L+t}+R_{L+t})$$
$$+(Z_{L+t}+R_{L+t})\phi(Z_{L+t}+R_{L+t})-(Z_{L+t}^2+1)\Phi(Z_{L+t})-Z_{L+t}\phi(Z_{L+t})-R_{L+\tau}(2Z_{L+t}+R_{L+t})\}dt. \quad (11)$$

Thus, a closed-form expression for average total cost model under Normal distributed demand is now fully determined as the sum of (9) and (11).

In order to apply the Newsboy-styled condition we also need to model $a(R,Q,T)$. But, this is nearly identical to the ordering probability P_o, so it can be modeled analogously. Using identical steps, we finally obtain:

$$a(R,Q,T)=\int_0^T\frac{1}{TR_{L+t}}\{\phi(Z_{L+t}+R_{L+t})+(Z_{L+t}+R_{L+t})\Phi(Z_{L+t}+R_{L+t})-\phi(Z_{L+t})-Z_{L+t}\Phi(Z_{L+t})\}dt \quad (12)$$

which, determines a closed-form expression for the α-measure under the conditions considered.

References

1. Anderson E.J. and Lagodimos A.G., (1989). Service levels in single-stage MRP systems with demand uncertainty. *Engineering Costs and Production Economics*, 17, 125–133.
2. Axsater S. and Rosling K., (1994). Multi-level production-inventory control - Material requirements planning or reorder point policies. *European Journal of Operational Research*, 78, 405–412.
3. Federgruen A. and Zipkin P., (1985). Computing optimal (s, S) policies in inventory systems with continuous demands. *Advances in Applied Probability*, 17, 424–442.
4. Hadley G. and Whitin T.M., (1961). A family of inventory models. *Management Science*, 7, 351–371.
5. Hadley G. and Whitin T.M., (1963). *Analysis of Inventory Systems* Prentice-Hall, New Jersey.
6. Larsen C. and Kiesmuller G.P., (2007). Developing closed-form cost expression for an policy where the demand process is generalized Erlang. *Operations Research Letters*, 35, 567–572.

7. Lee H.L. and Nahmias S., (1993). Single-product, single-location models. In: Graves S.C., Rinnoy Kan A.H.G. and Zipkin P.H. (editors) Logistics of Production and Inventory. Elsevier, Amsterdam, 3–55.

8. Li X. and Sridharan V., (2008) Characterizing order processes of using (R, nQ) inventory policies in supply chains. *Omega*, 36, 1096–1104.

9. Morse P.M., (1959) Solutions of a class of discrete time inventory problems. *Operations Research*, 7, 67–78.

10. Rao U.S., (2003) Properties of the periodic review (R, T) inventory control policy for stationary, stochastic demand. *Manufacturing and Service Operations Management*, 5, 37–53.

11. Serfozo R and Stidham S (1978), Semi-stationary Clearing Process, *Stochastic Process. Appl*, 6, 165-178.

12. Silver E.A., Pyke D.F. and Peterson R., (1998) *Inventory Management and Production Planning and Scheduling*. John Willey and Sons, New York.

13. Veinott A.F., (1965) The optimal inventory policy for batch ordering. *Operations Research*, 13, 424–432.

14. Veinott A.F., (1966) The status of mathematical inventory theory. *Management Science*, 12 745–777.

15. Veinott A.F. and Wagner H.M., (1965) Optimal (s, S) inventory policies. *Management Science*, 11, 525–552.

16. Zheng Y.S (1992). On properties of stochastic inventory systems. *Management Science*, 18, 87-103

17. Zheng Y.-S. and Chen F., (1992) Inventory policies with quantized ordering. *Naval Research Logistics*, 39, 285–305.

18. Zipkin, P.H. (1986) Inventory service-level measures: convexity and approximations. *Management Science*, 32, 975-981.

19. Zipkin, P.H., (2000) *Foundations of Inventory Management*. McGraw-Hill, New York.

EXISTENCE OF SOLUTIONS FOR A BOUNDARY VALUE PROBLEM ON AN INFINITE INTERVAL

KYRIAKOS G. MAVRIDIS AND PANAGIOTIS CH. TSAMATOS

ABSTRACT. Based on a fixed point theorem due to Avery and Henderson, we prove that a second order boundary value problem has at least two positive solutions.

1. INTRODUCTION

Some of the most widely used theorems guaranteeing the existence of one or multiple fixed points are the ones due to Krasnoselskii [19], Leggett and Williams [10], and Avery and Henderson [3]. Among the latest additions to this series of theorems are the ones due to Avery, Henderson and O'Regan [4, 5, 6]. An innovating attempt to unify all the results mentioned above, carried out by Kwong, can be found in [9]. Roughly speaking, the essence of all these theorems is to generalize the Intermediate Value Theorem for real functions of one real variable to function spaces, which are Banach spaces of infinite dimensions. One very important aspect of this generalization is to properly transfer the meaning of the closed interval of the real line to such spaces. An excellent discussion on this subject can be found in [2, 9, 19].

This paper is a sequel of [15]. The main result presented in [15] is based on the Krasnoselskii Fixed Point Theorem and provides conditions which guarantee the existence of at least one nonnegative solution for the boundary value problem studied therein. Our goal in this paper is to achieve multiple solutions for the ordinary version of the same boundary value problem. To do this, we use a fixed point theorem due to Avery and Henderson. This theorem, apart from guarantying the existence of two fixed points, provides some additional information about them, which varies depending on the way it is used. Here, we obtain upper or lower boundaries for the values of these fixed points at two predefined points of their domain.

2000 *Mathematics Subject Classification.* Primary: 34B40, Secondary: 34K10, 34B18, 47H10.

Key words and phrases. Avery-Henderson fixed point theorem, boundary value problem on the half-line, multiple positive solutions, second-order differential equations.

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^+ := [0, +\infty)$. Also, for any interval $I \subseteq \mathbb{R}$ and any set $S \subseteq \mathbb{R}$, by $C(I, S)$ we denote the set of all continuous functions defined on I, which have values in S. Consider the second order nonlinear differential equation

$$x''(t) + f(t, x(t)) = 0, \quad t \in \mathbb{R}^+ \tag{1.1}$$

along with the initial condition

$$x(0) = 0 \tag{1.2}$$

and the boundary condition

$$\lim_{t \to +\infty} x'(t) = \xi \tag{1.3}$$

where f is a real valued function defined on the set $\mathbb{R}^+ \times \mathbb{R}$, which is increasing with respect to its second variable, nonnegative and continuous, and ξ is a nonnegative real number.

2. Preliminaries and Lemmas

Definition 2.1. A function $x \in C(\mathbb{R}^+, \mathbb{R})$ is a solution of the boundary value problem $(1.1) - (1.3)$ if x is twice continuously differentiable and satisfies equation (1.1) and the boundary condition (1.3).

Definition 2.2. Let E be a real Banach space. A cone in E is a nonempty, closed set $P \subseteq E$ such that

(*i*) $\kappa u + \lambda v \in P$ for all $u, v \in P$ and all $\kappa, \lambda \geq 0$,
(*ii*) $u, -u \in P$ implies $u = 0$.

Definition 2.3. Let P be a cone in a real Banach space E. A functional $\psi : P \to E$ is said to be increasing on P if $\psi(x) \leq \psi(y)$, for any $x, y \in P$ with $x \leq y$, where $\leq$ is the partial ordering induced to the Banach space by the cone P, i.e.

$$x \leq y \quad \text{if and only if} \quad y - x \in P.$$

Definition 2.4. Let ψ be a nonnegative functional on a cone P. For each $d > 0$, we denote by $P(\psi, d)$ the set

$$P(\psi, d) := \{x \in P : \psi(x) < d\}.$$

The results of this paper are based on the following fixed point theorem, due to Avery and Henderson [3].

Theorem 2.5. *Let P be a cone in a real Banach space E. Let α and γ be increasing, nonnegative, continuous functionals on P, and let θ be a nonnegative functional on P with $\theta(0) = 0$ such that, for some $c > 0$ and $\Theta > 0$,*

$$\gamma(x) \leq \theta(x) \leq \alpha(x) \quad \textit{and} \quad \|x\| \leq \Theta\gamma(x),$$

for all $x \in \overline{P(\gamma, c)}$. *Suppose there exists a completely continuous operator* $A : \overline{P(\gamma, c)} \to P$ *and real constants* a, b *with* $0 < a < b < c$, *such that*

$$\theta(\lambda x) \leq \lambda \theta(x), \quad \text{for} \quad 0 \leq \lambda \leq 1 \quad \text{and} \quad x \in \partial P(\theta, b),$$

and either

(*i*) $\gamma(Ax) > c$, *for all* $x \in \partial P(\gamma, c)$,
(*ii*) $\theta(Ax) < b$, *for all* $x \in \partial P(\theta, b)$,
(*iii*) $P(\alpha, a) \neq \emptyset$, *and* $\alpha(Ax) > a$, *for all* $x \in \partial P(\alpha, a)$

or

(*i*) $\gamma(Ax) < c$, *for all* $x \in \partial P(\gamma, c)$,
(*ii*) $\theta(Ax) > b$, *for all* $x \in \partial P(\theta, b)$,
(*iii*) $P(\alpha, a) \neq \emptyset$, *and* $\alpha(Ax) < a$, *for all* $x \in \partial P(\alpha, a)$.

Then A *has at least two fixed points* x_1 *and* x_2 *belonging to* $\overline{P(\gamma, c)}$ *such that*

$$a < \alpha(x_1), \quad \text{with} \quad \theta(x_1) < b,$$

and

$$b < \theta(x_2), \quad \text{with} \quad \gamma(x_2) < c.$$

3. Main Results

Let $BC(\mathbb{R}^+, \mathbb{R})$ be the Banach space of all bounded continuous real valued functions on the interval $\mathbb{R}^+$, endowed with the sup-norm $\|\cdot\|$ defined by

$$\|u\| := \sup_{t \geq 0} |u(t)|, \quad \text{for} \quad u \in BC(\mathbb{R}^+, \mathbb{R}).$$

Definition 3.1. A set U of real valued functions defined on the interval $\mathbb{R}^+$ is called equiconvergent at ∞ if all functions in U are convergent in $\mathbb{R}$ at the point ∞ and, in addition, for each $\epsilon > 0$, there exists $T \equiv T(\epsilon) > 0$ such that, for all functions $u \in U$, it holds

$$|u(t) - \lim_{s \to \infty} u(s)| < \epsilon, \quad \text{for every} \quad t \geq T.$$

Lemma 3.2. *Let* U *be an equicontinuous and uniformly bounded subset of the Banach space* $BC(\mathbb{R}^+, \mathbb{R})$. *If* U *is equiconvergent at* ∞, *it is also relatively compact.*

Let

$$E = \{y \in C(\mathbb{R}^+, \mathbb{R}) : \ y(t) = O(t) \ \text{ for } \ t \to +\infty\}.$$

The set E is a real Banach space endowed with the norm $\|\cdot\|_E$, defined by

$$\|y\|_E := \sup_{t\geq 0} \frac{|y(t)|}{t+1}, \quad \text{for every} \quad y \in E.$$

Also, we define the following set K, which is a cone in E

$$K := \{x \in E: \ x(0) = 0, \ x(t) \geq \min\{t,1\}\|x\|_E, \text{ for } t \in \mathbb{R}^+, \\ \text{and } x \text{ is nondecreasing}\}.$$

Let

$$0 < r_1 \leq r_2 \leq r_3 \leq 1$$

and consider the following functionals

$$\gamma(x) = x(r_1), \quad x \in K$$
$$\theta(x) = x(r_2), \quad x \in K$$

and

$$\alpha(x) = x(r_3), \quad x \in K.$$

It is easy to see that α, γ are nonnegative, increasing and continuous functionals on K, θ is nonnegative on K and $\theta(0) = 0$. Also, it is straightforward that

$$\gamma(x) \leq \theta(x) \leq \alpha(x),$$

since $x \in K$ is nondecreasing on $\mathbb{R}^+$. Furthermore, for any $x \in K$, we have

$$\gamma(x) = x(r_1) \geq r_1\|x\|_E,$$

so

$$\|x\|_E \leq \frac{1}{r_1}\gamma(x), \quad x \in K.$$

Additionally, by the definition of θ it is obvious that

$$\theta(\lambda x) = \lambda\theta(x), \quad 0 \leq \lambda \leq 1, \quad x \in K.$$

At this point, we state the following assumptions.

($\mathbf{H_1}$) There exists $M > \xi$, a continuous function $u : \mathbb{R}^+ \to \mathbb{R}^+$ and a nondecreasing function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(t,y) \leq u(t)L\left(\frac{y}{1+t}\right), \quad t \in \mathbb{R}^+, \quad y \in \mathbb{R}^+$$

and also

$$\xi r_2 + L(M)\left[\int_0^{r_2} su(s)ds + r_2\int_{r_2}^{\infty} u(s)ds\right] < Mr_2.$$

(H_2) There exist a constant $\delta \in (0,1]$, a continuous function $v : \mathbb{R}^+ \to \mathbb{R}^+$ and a nondecreasing function $w : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(t,y) \geq v(t)w(y), \quad t \in [\delta, +\infty), \quad y \in \mathbb{R}^+.$$

(H_3) There exist $\rho_1, \rho_3 > 0$ such that

$$\frac{\rho_i}{\delta}(r_i+1) < \xi r_i + w(\rho_i)\left[\int_{[0,r_i]\cap[\delta,+\infty)} sv(s)ds + r_i \int_{[r_i,\infty)\cap[\delta,+\infty)} v(s)ds\right],$$

for $i = 1, 3$ and

$$\frac{\rho_3}{\delta}(r_3+1) < Mr_2 < \frac{\rho_1}{\delta}(r_1+1).$$

Lemma 3.3. *Suppose that assumption (H_1) holds and let $\epsilon > 0$. A function $x \in \overline{K(\gamma,\epsilon)}$ is a solution of the boundary value problem* (1.1)–(1.3) *if and only if x is a fixed point of the operator $A : \overline{K(\gamma,\epsilon)} \to C(\mathbb{R}^+, \mathbb{R})$, defined by the formula*

$$Ay(t) := \xi t + \int_0^\infty \min\{t,s\} f(s,y(s))ds, \quad \textit{for every } t \in \mathbb{R}^+, \tag{3.1}$$

or, equivalently,

$$Ay(t) := \xi t + \int_0^t sf(s,y(s))ds + t\int_t^\infty f(s,y(s))ds, \quad \textit{for every } t \in \mathbb{R}^+. \tag{3.2}$$

Proof. First of all, we will show that operator A is well defined. Indeed, for any $\epsilon > 0$ and any $x \in \overline{K(\gamma,\epsilon)}$ we have

$$r_1\|x\|_E \leq x(r_1) \leq \epsilon$$

and

$$\|x\|_E \leq \frac{\epsilon}{r_1}.$$

Also, for every $t \in \mathbb{R}^+$, it holds that

$$\frac{x(t)}{1+t} \leq \sup_{\sigma\in\mathbb{R}^+} \frac{x(\sigma)}{1+\sigma} = \|x\|_E \leq \frac{\epsilon}{r_1}.$$

Consequently, for any $t \in \mathbb{R}^+$, using assumption (H_1), we have

$$f(t,x(t)) \leq u(t)L\left(\frac{x(t)}{1+t}\right) \leq u(t)L\left(\frac{\epsilon}{r_1}\right),$$

therefore,

$$\int_0^\infty f(s,x(s))ds \leq \int_0^\infty u(s)L\left(\frac{\epsilon}{r_1}\right)ds = L\left(\frac{\epsilon}{r_1}\right)\int_0^\infty u(s)ds < \infty.$$

Hence, the formula of operator A makes sense for any $x \in \overline{K(\gamma,\epsilon)}$.

For the rest of the proof, see [16]. □

Lemma 3.4. *Suppose that assumption (H_1) holds. Then, the operator A is completely continuous and, for every $\epsilon > 0$, maps $\overline{K(\gamma,\epsilon)}$ into K.*

Proof. First, we will show that A maps $\overline{K(\gamma,\epsilon)}$ into K. Let $x \in \overline{K(\gamma,\epsilon)}$. Then obviously $Ax(t) \geq 0$ for every $t \in \mathbb{R}^+$, and $Ax(0) = 0$. Additionally,

$$(Ax)'(t) = \xi + \int_t^\infty f(s,x(s))ds \geq 0, \quad \text{for every} \quad t \in \mathbb{R}^+.$$

Next, we observe that, for any nonnegative real numbers t and σ, it holds

$$t \geq \begin{cases} \frac{t}{\sigma+1}\sigma & \text{for } t \in [0,1], \\ \frac{1}{\sigma+1}\sigma & \text{for } t \in [1,\infty). \end{cases}$$

That is

$$t \geq \frac{\min\{t,1\}}{\sigma+1}\sigma, \quad \text{for every } t \geq 0 \text{ and } \sigma \geq 0. \tag{3.3}$$

Moreover, it is not difficult to verify that, if t, s, σ are arbitrary nonnegative real numbers, then

$$\min\{t,s\} \geq \begin{cases} \frac{t}{\sigma+1}\min\{\sigma,s\} & \text{for } t \in [0,1], \\ \frac{1}{\sigma+1}\min\{\sigma,s\} & \text{for } t \in [1,\infty). \end{cases}$$

Namely, we have

$$\min\{t,s\} \geq \frac{\min\{t,1\}}{\sigma+1}\min\{s,\sigma\}, \quad \text{for every } t,s,\sigma \geq 0. \tag{3.4}$$

Since the function f is nonnegative and using (3.3) and (3.4), we obtain, for every $t \geq 0$ and $\sigma \geq 0$,

$$\begin{aligned} Ax(t) =& \xi t + \int_0^\infty \min\{t,s\} f(s,x(s))ds \\ \geq& \xi\frac{\min\{t,1\}}{\sigma+1}\sigma + \frac{\min\{t,1\}}{\sigma+1}\int_0^\infty \min\{\sigma,s\} f(s,x(s))ds \\ =& \min\{t,1\}\left\{\frac{1}{\sigma+1}\left(\xi\sigma + \int_0^\infty \min\{\sigma,s\} f(s,x(s))ds\right)\right\} \\ =& \min\{t,1\}\frac{Ax(\sigma)}{\sigma+1}. \end{aligned}$$

Therefore,

$$Ax(t) \geq \min\{t,1\}\sup_{\sigma\geq 0}\frac{Ax(\sigma)}{\sigma+1}, \quad \text{for every } t \geq 0,$$

i.e.

$$Ax(t) \geq \min\{t,1\}\|Ax\|_E, \quad \text{for every } t \geq 0.$$

Consequently $Ax \in K$.

Also, similarly to [16], we can prove that $A(\overline{K(\gamma, c)})$ is relatively compact and A is continuous. So, we have proved that the operator A is completely continuous. □

Theorem 3.5. *Suppose that assumptions* (H_1)-(H_3) *hold. Then the boundary value problem* (1.1)-(1.3) *has at least two nondecreasing, concave and positive on* $\mathbb{R}^+$ *solutions* $x, \tilde{x}$ *such that*

$$x(r_3) > \frac{\rho_3}{\delta}(r_3 + 1), \quad x(r_2) < Mr_2$$

and

$$\tilde{x}(r_1) < \frac{\rho_1}{\delta}(r_1 + 1), \quad \tilde{x}(r_2) > Mr_2.$$

Proof. Set $a = \frac{\rho_3}{\delta}(r_3 + 1)$, $b = Mr_2$ and $c = \frac{\rho_1}{\delta}(r_1 + 1)$. From Lemma 3.4, we have that A is a completely continuous operator, which maps $\overline{K(\gamma, c)}$ into K.

Now, let $x \in \partial K(\gamma, c)$. Then $\gamma(x) = x(r_1) = c$, so

$$\|x\|_E \geq \frac{c}{r_1 + 1}. \tag{3.5}$$

Having in mind assumption (H_2), we get

$$\begin{aligned}
\gamma(Ax) &= Ax(r_1) \\
&= \xi r_1 + \int_0^{r_1} sf(s, x(s))ds + r_1 \int_{r_1}^{\infty} f(s, x(s))ds \\
&\geq \xi r_1 + \int_{[0,r_1]\cap[\delta,+\infty)} sf(s, x(s))ds + r_1 \int_{[r_1,\infty)\cap[\delta,+\infty)} f(s, x(s))ds \\
&\geq \xi r_1 + \int_{[0,r_1]\cap[\delta,+\infty)} sv(s)w(x(s))ds + r_1 \int_{[r_1,\infty)\cap[\delta,+\infty)} v(s)w(x(s))ds \\
&\geq \xi r_1 + \int_{[0,r_1]\cap[\delta,+\infty)} sv(s)w(x(\delta))ds + r_1 \int_{[r_1,\infty)\cap[\delta,+\infty)} v(s)w(x(\delta))ds.
\end{aligned}$$

So, since $x \in K$, we have

$$\begin{aligned}
\gamma(Ax) &\geq \xi r_1 + \int_{[0,r_1]\cap[\delta,+\infty)} sv(s)w(\delta\|x\|_E)ds + r_1 \int_{[r_1,\infty)\cap[\delta,+\infty)} v(s)w(\delta\|x\|_E)ds \\
&= \xi r_1 + w(\delta\|x\|_E)\left[\int_{[0,r_1]\cap[\delta,+\infty)} sv(s)ds + r_1 \int_{[r_1,\infty)\cap[\delta,+\infty)} v(s)ds\right].
\end{aligned}$$

At this point, we use (3.5) and we get

$$\begin{aligned}\gamma(Ax) &\geq \xi r_1 + w(\delta\frac{c}{r_1+1})\left[\int_{[0,r_1]\cap[\delta,+\infty)} sv(s)ds + r_1\int_{[r_1,\infty)\cap[\delta,+\infty)} v(s)ds\right]\\ &= \xi r_1 + w(\rho_1)\left[\int_{[0,r_1]\cap[\delta,+\infty)} sv(s)ds + r_1\int_{[r_1,\infty)\cap[\delta,+\infty)} v(s)ds\right].\end{aligned}$$

Using hypothesis ($\mathbf{H_3}$), we conclude that

$$\gamma(Ax) > \frac{\rho_1}{\delta}(r_1+1),$$

so condition (i) of Theorem 2.5 is satisfied.

Now let $x \in \partial K(\theta, b)$. Then $\theta(x) = x(r_2) = b$, so since $x \in K$, we have

$$\|x\|_E \leq \frac{x(r_2)}{r_2} = \frac{b}{r_2} = M.$$

Consequently, by assumption ($\mathbf{H_1}$), we have

$$\begin{aligned}\theta(Ax) &= Ax(r_2)\\ &= \xi r_2 + \int_0^{r_2} sf(s,x(s))ds + r_2\int_{r_2}^{\infty} f(s,x(s))ds\\ &\leq \xi r_2 + \int_0^{r_2} su(s)L\left(\frac{x(s)}{1+s}\right)ds + r_2\int_{r_2}^{\infty} u(s)L\left(\frac{x(s)}{1+s}\right)ds\\ &\leq \xi r_2 + \int_0^{r_2} su(s)L(M)ds + r_2\int_{r_2}^{\infty} u(s)L(M)ds\\ &= \xi r_2 + L(M)\left[\int_0^{r_2} su(s)ds + r_2\int_{r_2}^{\infty} u(s)ds\right].\end{aligned}$$

So

$$\theta(Ax) \leq Mr_2 = b,$$

which means that condition (ii) of Theorem 2.5 is satisfied.

Now, define the function $y : \mathbb{R}^+ \to \mathbb{R}$ with $y(t) = \frac{a}{2}$. Then, it is obvious that $\alpha(y) = \frac{a}{2} < a$, so $K(\alpha, a) \neq \emptyset$. Also, since $\alpha(x) = x(r_3) = a$, we have $\frac{x(r_3)}{r_3+1} = \frac{a}{r_3+1}$, so

$$\|x\|_E \geq \frac{a}{r_3+1}. \tag{3.6}$$

As in the case of the functional γ above, we get

$$\begin{aligned}\alpha(Ax) &= Ax(r_3)\\ &\geq \xi r_3 + \int_{[0,r_3]\cap[\delta,+\infty)} sv(s)w(x(\delta))ds + r_3\int_{[r_3,\infty)\cap[\delta,+\infty)} v(s)w(x(\delta))ds.\end{aligned}$$

So, since $x \in K$, we have

$$\alpha(Ax) \geq \xi r_3 + w(\delta\|x\|_E)\left[\int_{[0,r_3]\cap[\delta,+\infty)} sv(s)ds + r_3 \int_{[r_3,\infty)\cap[\delta,+\infty)} v(s)ds\right]$$

and using (3.6) we get

$$\alpha(Ax) \geq \xi r_3 + w(\rho_3)\left[\int_{[0,r_3]\cap[\delta,+\infty)} sv(s)ds + r_3 \int_{[r_3,\infty)\cap[\delta,+\infty)} v(s)ds\right].$$

Therefore, by hypothesis (H_3), we conclude that

$$\alpha(Ax) > \frac{\rho_3}{\delta}(r_3 + 1),$$

so condition (*iii*) of Theorem 2.5 is satisfied.

At this point, we apply Theorem 2.5 to obtain that operator A has at least two fixed points x and $\tilde{x}$ belonging to $\overline{K(\gamma, c)}$ such that

$$x(r_3) > \frac{\rho_3}{\delta}(r_3 + 1), \quad x(r_2) < Mr_2$$

and

$$\tilde{x}(r_1) < \frac{\rho_1}{\delta}(r_1 + 1), \quad \tilde{x}(r_2) > Mr_2.$$

This concludes the proof. □

References

[1] R. P. Agarwal and D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer Academic publisher, 2001.

[2] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.* 18 (1976), no. 4, 620–709

[3] R. I. Avery and J. Henderson, *Two positive fixed points of nonlinear operators on ordered Banach spaces*, Comm. Appl. Nonlinear Anal., Vol. 8 (2001), 27–36.

[4] R. Avery, J. Henderson and D. O'Regan, Four functionals fixed point theorem, *Math. Comput. Modelling* 48 (2008), no. 7-8, 1081–1089.

[5] R. Avery, J. Henderson and D. O'Regan, Functional compression-expansion fixed point theorem, *Electron. J. Differential Equations* 2008, No. 22, 12 pp.

[6] R. Avery, J. Henderson and D. O'Regan, Six functionals fixed point theorem, *Commun. Appl. Anal.* 12 (2008), no. 1, 69–81.

[7] P. W. Eloe, L. J. Grimm and J. D. Mashburn, A boundary value problem on an unbounded domain, *Differential Equations Dynam. Systems*, 8, No. 2 (2000), 125–140.

[8] D. Guo, Second order integro-differential equations of Voltera type on unbounded domains in Banach spaces, *Nonlinear Anal.*, 41 (2000), 465–476.

[9] M. K. Kwong, On Krasnoselskii's cone fixed point theorem, *Fixed Point Theory Appl.* 2008, Art. ID 164537, 18 pp.

[10] R. W. Leggett and L. R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, *Indiana Univ. Math. J.* 28 (1979), no. 4, 673–688.

[11] H. Lian, H. Pang and W. Ge, Triple positive solutions for boundary value problems on infinite intervals, *Nonlinear Anal.* 67 (2007), no. 7, 2199–2207.

[12] Y. Liu, Boundary value problems for second order differential equations on unbounded domain, *Acta Anal. Funct. Appl.* 4 (3) (2002) 211–216.

[13] Y. Liu, Boundary value problems for second order differential equations on unbounded domains in a Banach space, *Appl. Math. Comput.*, 135 (2003), 569–583.

[14] Y. Liu, Existence and unboundedness of positive solutions for singular boundary value problems on half-line, *Appl. Math. Comput.*, 144 (2003), 543–556.

[15] K. G. Mavridis, Ch. G. Philos and P. Ch. Tsamatos, Multiple positive solutions for a second order delay boundary value problem on the half-line, *Ann. Polon. Math.* 88 (2006), no. 2, 173–191.

[16] K. G. Mavridis, Ch. G. Philos and P. Ch. Tsamatos, Existence of solutions of a boundary value problem on the half-line to second order nonlinear delay differential equations, *Arch. Math. (Basel)* 86 (2006), no. 2, 163–175.

[17] K. G. Mavridis and P. Ch. Tsamatos, Positive solutions for first order nonlinear functional boundary value problems on infinite intervals, *Electron. J. Qual. Theory Differ. Equ.* 2004, No. 8, 18 pp.

[18] B. Yan, Multiple unbounded solutions of boundary value problems for second-order differential equations on the half-line, *Nonlinear Anal.*, 51 (2002), 1031–1044.

[19] E. Zeidler, *Nonlinear functional analysis and its applications. I. Fixed-point theorems. Translated from the German by Peter R. Wadsack*, Springer-Verlag, New York, 1986.

[20] M. Zima, On positive solution of boundary value problems on the half-line, *J. Math. Anal. Appl.* 259 (2001) 127–136.

Department of Mathematics, University of Ioannina, P. O. Box 1186, 451 10 Ioannina, Greece

E-mail address: `kmavride@otenet.gr (K.G. Mavridis), ptsamato@uoi.gr (P.Ch. Tsamatos)`

C^1- APPROXIMATE SOLUTIONS OF SECOND ORDER SINGULAR ORDINARY DIFFERENTIAL EQUATIONS

GEORGE L. KARAKOSTAS

ABSTRACT. In this work a new method is developed to obtain C^1-approximate solutions of initial and boundary value problems generated from a one-parameter second order singular ordinary differential equation. Information about the order of approximation is also given by introducing the so called *growth index* of a function. Conditions are given for the existence of such approximations for initial and boundary value problems of several kinds. Examples associated with the corresponding graphs of the approximate solutions, for some values of the parameter, are also given.

CONTENTS

2000 *Mathematics Subject Classification.* 34A45; 34A12, 34A25, 34B99.

Key words and phrases. One-parameter second order ordinary differential equation; growth index of a function; Approximate solutions; Initial value problems; Boundary value problems;

March 21, 2010.

1. Introduction

A one-parameter perturbation singular problem associated with a second order ordinary differential equation is a problem whose the solutions behave nonuniformly near the initial (or the boundary) values, as the parameter approaches extreme levels. In this work we develop a new method to obtain approximate solutions of some problems of this kind. It is well known that under such a limiting process two situations may occur:

i) The limiting position of the system exists, thus one can talk about the continuous or discontinuous dependence of the solutions on the parameter.

Consider, for instance, the following one-parameter scalar autonomous Caushy problem

$$x'' + f(x,p) = 0, \;\; x(0) = \alpha, \;\; x'(0) = \beta,$$

when the parameter p takes large values (and tends to $+\infty$). Under the assumption that f satisfies some monotonicy conditions and it approaches a certain function g as the parameter p tends to $+\infty$, a geometric argument is used in the literature (see, e.g., Elias and Gingold [7]) to show, among others, that if the initial values lie in a suitable domain on the plane, then the solution approximates (in the C^1-sense) the corresponding solution of the limiting equation. The same behavior have the periods (in case of periodic solutions) and the escape times (in case of non-periodic solutions). Donal O' Regan in his informative book [15], p. 14, presents a problem involving a second order differential equation, when the boundary conditions are of the form $y(0) = a$ (fixed) and $y(1) = \frac{a}{n}$, when n is large enough. It is shown that for a delay equation of the form

$$\varepsilon\dot{x}(t) + x(t) = f(x(t-1)),$$

when f satisfies some rather mild conditions, there exists a periodic solution which is close to the square wave corresponding to the limiting (as $\varepsilon \to 0^+$) difference equation:

$$x(t) = f(x(t-1)).$$

Similarly, as it is shown in Ch. 10 of the book of Ferdinand Verhulst [22], the equation

$$x'' + x = \varepsilon f(x, x', \varepsilon), \;\; (x, x') \in D \subseteq \mathbb{R}^2 \tag{1.1}$$

($\varepsilon > 0$ and small) associated with the initial conditions

$$x(0) = a(\varepsilon), \;\; x'(0) = 0,$$

under some conditions on f, has a periodic solution $x(t;\varepsilon)$ satisfying

$$\lim_{\varepsilon\to 0+} x(t;\varepsilon) = a(0)\cos t.$$

Notice that the limiting value $a(0)\cos t$ is the solution of (1.1) when $\varepsilon = 0$.

ii) There exist some coefficients of the system which vanish, or tend to infinity, as the parameter approaches a liming value. In this case we can not formulate a limiting equation; however we have an asymptotic approximate system for values of the parameter which are close to the limiting value. The advantage of this situation is that in many circumstances it is possible to have information on the solutions of the limiting systems and, moreover, to compute (in closed form) the so-called approximate solutions.

A simple prototype of this situation is, for instance, the differential equation

$$\varepsilon\frac{d^2u}{dt^2} + 2\frac{du}{dt} + u = 0, \quad t > 0,$$

subject to the initial values

$$u(0) = a, \quad \frac{du}{dt} = b + \frac{\gamma}{\varepsilon}, \tag{1.2}$$

discussed in the literature and especially in the classic detailed book due to Donald R. Smith [19], p. 134. Here the parameter ε is small enough and it approaches zero.

A more general situation, which we will discuss later in Section 5, is an equation of the form

$$x'' + [a_1(t) + a_2(t)p^{\nu}]x' + [b_1(t) + b_2(t)p^{\mu}]x + a_0 p^m x\sin(x) = 0, \quad t > 0 \tag{1.3}$$

associated with the initial values

$$x(0;p) = \delta_1 + \delta_2 p^{\sigma}, \quad x'(0;p) = \eta_1 + \eta_2 p^{\tau}. \tag{1.4}$$

The entities μ, ν, m, σ and τ are real numbers and p is a large parameter.

The previous two problems have the general form

$$x''(t) + a(t;p)x'(t) + b(t;p)x(t) + f(t, x(t); p) = 0, \quad t > 0, \tag{1.5}$$

where the parameter p is large enough, while the initial values are of the form

$$x(0;p) = x_0(p), \quad x'(0;p) = \bar{x}_0(p). \tag{1.6}$$

It is well known that the Krylov-Bogoliubov method was developed in the 1930's to handle situations described by second order ordinary differential equations of the form (1.5) motivated by problems in mechanics of the type generated by the Einstein equation for Mercury.

This approach, which was applied to various problems presented in [19], is based on the so called O'Malley [12], [13] and Hoppensteadt [8] method. According to this method (in case f does not depend on x) we seek an additive decomposition of the solution x of (1.5) in the form

$$x(t;p) \sim U(t;p) + U^*(\tau;p),$$

where $\tau := tp$ is the large variable and U, U^* are suitable functions, which are to be obtained in the form of asymptotic expansions, as

$$U(t;p) = \sum_{k=0}^{\infty} U_k(t)p^{-k}$$

and

$$U^*(t;p) = \sum_{k=0}^{\infty} U_k^*(t)p^{-k}.$$

After the coefficients U_k and U_k^* are determined we define the remainder

$$R_N := R_N(t;p)$$

by the relation

$$x(t;p) = \sum_{k=0}^{\infty}[U_k(t) + U_k^*(t)]p^{-k} + R_N(t;p)$$

and then obtain suitable C^1 estimates of R_N (see, [19], p. 146). This method is applied when the solutions admit initial values as in (1.2). For the general O'Malley-Hoppensteadt construction an analogous approach is followed elsewhere, see [19], p. 117. In the book due to R.E. O' Malley [14] an extended exhibition of the subject is given. The central point of the method is to obtain approximation of the solution, when the system depends on a small parameter tending to zero, (or equivalently, on a large parameter tending to $+\infty$). The small parameter ϵ is used in some of these cases and the functions involved are smooth enough to guarantee the existence and uniqueness of solutions.

In the literature one can find a great number of works dealing with singular boundary value problems, performing a set of different methods. For instance, the work due to Kadalbajoo and Patidar [10] presents a (good background and a very rich list of references on the subject, as well as a) deep survey of numerical techniques used in many circumstances to solve singularly perturbed ordinary differential equations. Also, in [21] a problem of the form

$$-\varepsilon u''(t) + p(t)u'(t) + q(t)u(t) = f(x), \quad u(a) = \alpha_0, \quad u(b) = \alpha_1,$$

is discussed, by using splines fitted with delta sequences as numerical strategies for the solution. See, also, [20]. A similar problem is discussed in [5], where the authors use a fourth-order finite-difference method. In [11] a problem of the form

$$\varepsilon y''(t) + [p(y(x))]' + q(x, y(x)) = r(x), \;\; y(a) = \alpha, \;\; y(b) = \beta,$$

is investigated by reducing it into an equivalent first order initial value problem and then by applying an appropriate non-linear one-step explicit scheme. In [17], where a problem of the form

$$\varepsilon y''(t) = f(x, y, x'), \;\; y(a) = y_a, \;\; y(b) = y_b,$$

is discussed, a smooth locally-analytical method is suggested. According to this method first the author considers nonoverlapping intervals and then linearize the ordinary differential equation around a fixed point of each interval. The method applies by imposing some continuity conditions of the solution at the two end points of each interval and of its first-order derivative at the common end point of two adjacent intervals.

A similar problem as above, but with boundary conditions of the form

$$y'(0) - ay(0) = A, \;\; y'(1) + by(1) = B,$$

is presented in [1], where a constructive iteration procedure is provided yielding an alternating sequence which gives pointwise upper and lower bounds on the solution.

The so called method of *small intervals* is used in [23], where the same problem as above is discussed but with impulses. In some other works, as e.g. [4], [2] (see also the references therein) two-point boundary value problems concerning third order differential equations are investigated, when the conditions depend on the (small) parameter ε. The methods used in these problems are mainly computational.

In this work our contribution to the subject is to give (assumptions and) information on the existence and the form of a C^1-approximate solution $\tilde{x}(t;p)$ of the ordinary differential equation (1.5), when the parameter p tends to $+\infty$, but by following a different approach: We suggest a smooth transformation of the time through which the equation (1.5) looks like a perturbation of an equation of the same order and with constant coefficients. The latter is used to get the approximate solution of the original equation without using the Sturm transformation. Furthermore, these arguments permit us to provide information on the estimates

$$x(t;p) - \tilde{x}(t;p)$$

and

$$\frac{d}{dt}\Big(x(t;p) - \tilde{x}(t;p)\Big),$$

as p tends to $+\infty$, uniformly for t in compact intervals. To handle the "size" of the approximation we introduce and use a kind of measure of boundedness of a function, which we term *the growth index.*

Our approach differs from that one used (recently) in [3] for the equation of the form

$$x'' + (p^2 q_1(t) + q_2(t))x = 0, \tag{1.7}$$

when p approaches $+\infty$. In [3] the authors suggest a method to approximate the solutions of (1.7) satisfying the boundary conditions of the form

$$x(0) = x_0, \ \ x(1) = mx(\xi). \tag{1.8}$$

To do that they provide an approximation of the equation, and then (they claim that) as the parameter p tends to $+\infty$, the solution of the old equation approaches the solution of the new one. And this fact is an implication of the following claim:

If a function $\delta(p)$, $p \geq 0$ satisfies $\delta(p) = o(p^{-2})$, as $p \to +\infty$, then the solution of the equation

$$v''(z;p) + v(z;p) = \delta(p)v(z;p),$$

approaches the solution of the equation

$$v''(z;p) + v(z;p) = 0.$$

However, as one can easily see, this is true only when $v(z;p) = O(p^r)$, as $p \to +\infty$, uniformly for all z, for some $r \in (0,2)$. Therefore in order to handle such cases more information on the solutions are needed.

This work is organized as follows:

In Section 2 we introduce the meaning of the growth index of a function and some useful characteristic properties of it. The basic assumptions of our problem and the auxiliary transformation of the original equation (1.5) is presented in Section 3, while in Sections 4 and 6 we give results on the existence of C^1-approximate solutions of the initial value problem (1.3)-(1.6). In Section 4 we consider equation (1.5) when the coefficient $b(t;p)$ takes (only) positive values and in Section 6 we discuss the case when $b(t;p)$ takes (only) negative values. Illustrative examples are given in Sections 5 and 7. Section 8 of the work is devoted to the approximate solutions of the boundary value problem

$$x''(t) + a(t;p)x'(t) + b(t;p)x(t) + f(t, x(t);p) = 0, \ \ t \in (0,1), \tag{1.9}$$

associated with the boundary conditions of Dirichlet type

$$x(0;p) = x_0(p), \ \ x(1;p) = x_1(p), \tag{1.10}$$

where the boundary values depend on the parameter p, as well. Here we use the (fixed point theorem of) Nonlinear Alternative to show the existence of solutions and then we present the approximate solutions. Some applications of these results are given in Section 9. In Section 10 we investigate the existence of C^1-approximate solutions of equation (1.9) associated with the boundary conditions (1.8). Again, the Nonlinear Alternative is used for the existence of solutions and then C^1-approximate solutions are given. An application of this result is given in the last section 11.

2. The growth index of a function

Before proceeding to the discussion of the main problem it is convenient to present some auxiliary facts about the growth of a real valued function f defined in a neighborhood of $+\infty$. For such a function we introduce an index, which, in a certain sense denotes the critical point at which the function stays in a real estate as the parameter tends to $+\infty$, relatively to a positive and unbounded function $E(\cdot)$. This meaning, which we term *the growth index* of f, will help us to calculate and better understand the approximation results. More facts about the growth index of functions will be published in a subsequent work.

All the (approximation) results of this work are considered with respect to a basic positive function $E(p)$, $p \geq 0$, as, e.g., $E(p) := \exp(p)$, or in general $E(p) := \exp^{(n)}(p)$, for all integers n. Here $\exp^{(0)}(p) := p$, and $\exp^{(-k)}(p) := \log^{(k)}(p)$, for all positive integers k. Actually, the function $E(p)$ denotes the level of convergence to $+\infty$ of a function h satisfying $h(p) = O((E(p))^{\mu})$, as $p \to +\infty$. The latter stands for the well known big-O symbol.

From now on we shall keep fixed such a function $E(p)$. To this function corresponds the set

$$\mathcal{A}_E := \{h : [0, +\infty) : \exists b \in \mathbb{R} : \limsup_{p \to +\infty} (E(p))^b |h(p)| < +\infty\}.$$

Then, for any $h \in \mathcal{A}_E$ we define the set

$$\mathcal{N}_E(h) := \{b \in \mathbb{R} : \limsup_{p \to +\infty} (E(p))^b |h(p)| < +\infty\}.$$

It is obvious that the set $\mathcal{N}_E(h)$ is a connected interval of the real line, whenever it is nonvoint[1]. In this case a very characteristic property of

[1]For instance, for the function $E(p) := p^2$ and a function like $h(p) := e^{\rho p}$, $\lambda > 0$ the set $\mathcal{N}_E(h)$ is empty.

the function $h \in \mathcal{A}_E$ is the quantity

$$\mathcal{G}_E(h) := \sup \mathcal{N}_E(h),$$

which we call *the growth index of h with respect to E.* To save space in the sequel the expression *with respect to E* will not be used.

The simplest case for the growth index can be met in case of the logarithm of the absolute value of an (entire complex valued function) of finite order. Indeed, if F is such a function, its order is defined as the least of all reals α such that

$$|F(z)| \leq \exp(|z|^\alpha),$$

for all complex numbers z. Now, the function $f(p) := \log|F(p+i0)|$ satisfies

$$\limsup_{p\to+\infty}(E(p))^b|f(p)| < +\infty$$

for all $b \leq -\alpha$, with respect to the level $E(p) := p$. Thus we have $\mathcal{G}_E(f) \geq -\alpha$.

More generally, the growth index of a function h such that $h(p) = O(p^k)$, as $p \to +\infty$, for some $k \in \mathbb{R}$, satisfies $\mathcal{G}_E(h) \geq -k$. Also, we observe that, if it holds

$$\mathcal{G}_E(h) > b,$$

then the function h satisfies

$$h(p) = O\Big([E(p)]^{-b}\Big), \quad \text{as } p \to +\infty,$$

or equivalently,

$$|h(p)| \leq K(E(p))^{-b},$$

for all p large enough and for some $K > 0$, not depending on p.

We present a list of characteristic properties of the growth index; some of them will be useful in the sequel.

Proposition 2.1. *If h_1, h_2 are elements of the class $\mathcal{A}_E$, then their product h_1h_2 also is an element of the same space $\mathcal{A}_E$ and moreover it holds*

$$\mathcal{G}_E(h_1h_2) \geq \mathcal{G}_E(h_1) + \mathcal{G}_E(h_2).$$

Proof. Given $h_1, h_2 \in \mathcal{A}_E$, take any b_1, b_2 such that $b_j < \mathcal{G}_E(h_j)$, $j = 1, 2$. Thus we have

$$\limsup_{p\to+\infty}(E(p))^{b_1}|h_1(p)| < +\infty \text{ and } \limsup_{p\to+\infty}(E(p))^{b_2}|h_2(p)| < +\infty$$

and therefore

$$\limsup_{p\to+\infty}(E(p))^{b_1+b_2}|h_1(p)h_2(p)| \le \limsup_{p\to+\infty}(E(p))^{b_1}|h_1(p)| \times \limsup_{p\to+\infty}(E(p))^{b_2}|h_2(p)| < +\infty.$$

This shows, first, that $h_1h_2 \in \mathcal{A}_E$ and, second, that $\mathcal{G}_E(h_1h_2) \ge b_1+b_2$. The latter implies that

$$\mathcal{G}_E(h_1h_2) \ge \mathcal{G}_E(h_1) + \mathcal{G}_E(h_2).$$

□

Lemma 2.2. *Consider the functions $h_1, h_2, \cdots, h_n$ in $\mathcal{A}_E$. Then, for all real numbers $a_j > 0$, the function $\sum_{j=1}^n a_jh_j$ belongs to $\mathcal{A}_E$ and moreover it satisfies*

$$\mathcal{G}_E\Big(\sum_{j=1}^n a_jh_j\Big) = \min\{\mathcal{G}_E(h_j) : j = 1, 2, \cdots, n\}. \tag{2.1}$$

Proof. The fact that $\sum_{j=1}^n a_jh_j$ is an element of $\mathcal{A}_E$ is obvious. To show the equality in (2.1), we assume that the left side of (2.1) is smaller than the right side. Then there is a real number N such that

$$\mathcal{G}_E\Big(\sum_{j=1}^n \alpha_jh_j\Big) < N < \min\{\mathcal{G}_E(h_j) : j = 1, 2, \cdots, n\}.$$

Thus, on one hand we have

$$\limsup_{p\to+\infty}\sum_{j=1}^n a_j\big(E(p)\big)^N|h_j(p)| = \limsup_{p\to+\infty}(E(p))^N\Big(\sum_{j=1}^n a_j|h_j(p)|\Big) = +\infty \tag{2.2}$$

and on the other hand it holds

$$\limsup_{p\to+\infty}(E(p))^N|h_j(p)| < +\infty, \quad j = 1, 2, ..., n.$$

The latter implies that

$$\limsup_{p\to+\infty}\sum_{j=1}^n a_j\big(E(p)\big)^N|h_j(p)| \le \sum_{j=1}^n a_j \limsup_{p\to+\infty}\big(E(p)\big)^N|h_j(p)| < +\infty,$$

contrary to (2.2).

If the right side of (2.1) is smaller than the left one, there is a real number N such that

$$\mathcal{G}_E\Big(\sum_{j=1}^{n} a_j h_j\Big) > N > \min\{\mathcal{G}_E(h_j) : j = 1, 2, \cdots, n\}.$$

Thus, on one hand we have

$$\limsup_{p\to+\infty}(E(p))^N \sum_{j=1}^{n} a_j |h_j(p)| < +\infty \tag{2.3}$$

and on the other hand it holds

$$\limsup_{p\to+\infty}(E(p))^N |h_{j_0}(p)| = +\infty,$$

for some $j_0 \in \{1, 2, ..., n\}$. The latter implies that

$$\limsup_{p\to+\infty}(E(p))^N \sum_{j=1}^{n} a_j |h_j(p)| \geq \limsup_{p\to+\infty} a_{j_0}(E(p))^N |h_{j_0}(p)| = +\infty,$$

contrary to (2.3). □

The growth index of a function denotes the way of convergence to zero at infinity of the function. Indeed, we have the following:

Proposition 2.3. *For a given function* $h : [r_0, +\infty) \to \mathbb{R}$ *it holds*

$$\mathcal{G}_E(h) = \sup\{r \in \mathbb{R} : \limsup_{p\to+\infty}(E(p))^r |h(p)| = 0\}.$$

Proof. If $b > \mathcal{G}_E(h)$, then

$$\limsup_{p\to+\infty}(E(p))^b |h(p)| = +\infty.$$

Thus, it is clearly enough to show that for any real b with $b < \mathcal{G}_E(h)$ it holds

$$\limsup_{p\to+\infty}(E(p))^b |h(p)| = 0.$$

To this end consider real numbers $b < b_1 < \mathcal{G}_E(h)$. Then we have

$$\limsup_{p\to+\infty}(E(p))^{b_1} |h(p)| =: K < +\infty$$

and therefore

$$\begin{aligned}\limsup_{p\to+\infty}(E(p))^b |h(p)| &= \limsup_{p\to+\infty}(E(p))^{(b-b_1)} \limsup_{p\to+\infty}(E(p))^{b_1} |h(p)| \\ &= \limsup_{p\to+\infty}(E(p))^{(b-b_1)} K = 0.\end{aligned}$$

□

In the sequel the choice of a variable t uniformly in compact subsets of a set U will be denoted by

$$t \in Co(U).$$

Especially we make the following:

Notation 2.4. *Let $H(t;p)$ be a function defined for $t \in S \subseteq \mathbb{R}$ and p large enough. In the sequel in case we write*

$$H(t;p) \simeq 0, \quad \text{as } p \to +\infty, \quad t \in Co(S),$$

we shall mean that given any compact set $I \subseteq S$ and any $\varepsilon > 0$ there is some $p_0 > 0$ such that

$$|H(t;p)| \leq \varepsilon,$$

for all $t \in I$ and $p \geq p_0$.

Also, keeping in mind Proposition 2.3 we make the following:

Notation 2.5. *Again, let $h(t;p)$ be a function defined for $t \in S \subseteq \mathbb{R}$ and p large enough. Writing*

$$\mathcal{G}_E(h(t;p)) \geq b, \quad t \in Co(S),$$

we shall mean that, for any $m < b$, it holds

$$(E(p))^m h(t;p) \simeq 0, \quad \text{as } p \to +\infty, \quad t \in Co(S).$$

3. Transforming equation (1.5)

In this section our purpose is to present a transformation of the one-parameter family of differential equations of the form (1.5), to a second order ordinary differential equation having constant coefficients.

Let $T_0 > 0$ be fixed and define $I := [0, T_0)$. Assume that the functions a, b, f are satisfying the following:

Condition 3.1. *For all large p the following statements are true:*

(1) The function $f(\cdot, \cdot; p)$ is continuous,

(2) $a(\cdot; p) \in C^1(I)$,

(3) There exists some $\theta > 0$ such that $|b(t;p)| \geq \theta$, for all t and all p large. Also assume that $b(\cdot;p) \in C^2(I)$ and $sign[b(t;p)] =: c =$ constant, for all $t \in I$.

The standard existence theory ensures that if Condition 3.1 holds, then equation (1.5) admits at least one solution defined on a (nontrivial) maximal interval of the form $[0, T) \subseteq [0, T_0)$.

To proceed, fix any $\hat{t} \in (0,T)$ and, for a moment, consider a strictly increasing one parameter C^2- mapping

$$v = v(t;p): \ [0,\hat{t}] \longrightarrow [0, v(\hat{t},p)] =: J$$

with $v(0;p) = 0$. Let $\phi(\cdot;p)$ be the inverse of $v(\cdot;p)$. These functions will be defined later. If $x(t;p)$, $t \in [0,\hat{t}]$ is a solution of (1.5), define the transformation

$$S_p: \ x(\cdot;p) \to S_p x(\cdot;p): \ Graph(x(\cdot;p))\Big(\subseteq C([0,\hat{t}],\mathbb{R})\Big) \to C(J,\mathbb{R}),$$

where

$$(S_p x(\cdot;p))(v) =: y(v;p) := \frac{x(t;p)}{Y(t;p)} = \frac{x(\phi(v;p);p)}{Y(\phi(v;p);p)}, \ v \in J. \tag{3.1}$$

Here $Y(\cdot;p)$, which will be specified later, is a certain C^2-function, depending on the parameter p. We observe that

$$x'(t;p) = Y'(t;p)y(v;p) + Y(t;p)v'(t;p)y'(v;p), \ t \in [0,\hat{t}]$$

and

$$\begin{aligned} x''(t;p = {} & Y''(t;p)y(v;p) + 2Y'(t;p)v'(t;p)y'(v;p) \\ & + Y(t;p)v''(t;p)y'(v;p) \\ & + Y(t;p)(v'(t;p))^2 y''(v;p), \ t \in [0,\hat{t}]. \end{aligned}$$

Then, equation (1.5) is transformed into the equation

$$y''(v;p) + A(t;p)y'(v;p) + B(t;p)y(v;p) + g(t;p) = 0, \ v \in J, \tag{3.2}$$

where the one-parameter functions A, B and g are defined as follows:

$$A(t;p) := \frac{2Y'(t;p)v'(t;p) + Y(t;p)v''(t;p) + a(t;p)Y(t;p)v'(t;p)}{Y(t;p)(v'(t;p))^2},$$

$$B(t;p) := \frac{Y''(t;p) + a(t;p)Y'(t;p) + b(t;p)Y(t;p)}{Y(t;p)(v'(t;p))^2},$$

$$g(t;p) := \frac{f(t, Y(t;p)y(v;p);p)}{Y(t;p)(v'(t;p))^2}.$$

We will specify the new functions v and Y. To get the specific form of the function $v(\cdot;p)$ we set

$$v'(t;p) = \sqrt{cb(t;p)}, \ t \in I, \tag{3.3}$$

where, recall that,

$$c = sign[b(t;p)], \ t \in I.$$

In order to have $v(t;p) \geq v(0;p) = 0$, it is enough to get

$$v(t;p) = \int_0^t \sqrt{cb(s;p)}ds, \ t \in [0,\hat{t}]. \tag{3.4}$$

Setting the coefficient $A(t;p)$ in (3.2) equal to zero, we obtain

$$2Y'(t;p)v'(t;p)+Y(t;p)v''(t;p)+a(t;p)Y(t;p)v'(t;p)=0,\quad t\in[0,\hat{t}], \tag{3.5}$$

which, due to (3.3), implies that

$$Y'(t;p)+\Big(\frac{b'(t;p)}{4b(t;p)}+\frac{a(t;p)}{2}\Big)Y(t;p)=0,\quad t\in[0,\hat{t}]. \tag{3.6}$$

We solve this equation, by integration and obtain

$$Y(t;p)=Y(0;p)\exp\Big(\int_0^t[-\frac{b'(s;p)}{4b(s;p)}-\frac{a(s;p)}{2}]ds\Big),$$

namely,

$$Y(t;p)=\Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}}\exp\Big(-\frac{1}{2}\int_0^t a(s;p)ds\Big),\quad t\in[0,\hat{t}], \tag{3.7}$$

where, without lost of generality, we have set $Y(0;p)=1$.

From (3.6) it follows that

$$\frac{Y'(t;p)}{Y(t;p)}=-\frac{b'(t;p)}{4b(t;p)}-\frac{a(t;p)}{2}, \tag{3.8}$$

from which we get

$$\begin{aligned}Y''(t;p)=&-Y'(t;p)\Big(\frac{b'(t;p)}{4b(t;p)}+\frac{a(t;p)}{2}\Big)\\&-Y(t;p)\Big(\frac{b(t;p)b''(t;p)-[b'(t;p)]^2}{4[b(t;p)]^2}+\frac{a'(t;p)}{2}\Big).\end{aligned} \tag{3.9}$$

Then, from relations (3.6), (3.8) and (3.9) we obtain

$$\begin{aligned}&Y''(t;p)+a(t;p)Y'(t;p)+b(t;p)Y(t;p)\\&=-Y'(t;p)\Big(\frac{b'(t;p)}{4b(t;p)}-\frac{a(t;p)}{2}\Big)\\&-Y(t;p)\Big(\frac{b(t;p)b''(t;p)-[b'(t;p)]^2}{4[b(t;p)]^2}+\frac{a'(t;p)}{2}-b(t;p)\Big)\\&=Y(t;p)\Big[\Big(\frac{b'(t;p)}{4b(t;p)}+\frac{a(t;p)}{2}\Big)\Big(\frac{b'(t;p)}{4b(t;p)}-\frac{a(t;p)}{2}\Big)\\&-\frac{b(t;p)b''(t;p)-(b'(t;p))^2}{4(b(t;p))^2}-\frac{a'(t;p)}{2}+b(t;p)\Big].\end{aligned}$$

Hence, the expression of the function B appeared in (3.2) takes the form

$$\begin{aligned}B(t;p) = & -\frac{1}{(v'(t;p))^2}\Big[\Big(\frac{b'(t;p)}{4b(t;p)}+\frac{a(t;p)}{2}\Big)\Big(\frac{b'(t;p)}{4b(t;p)}-\frac{a(t;p)}{2}\Big)\\ & -\frac{b(t;p)b''(t;p)-[b'(t;p)]^2}{4[b(t;p)]^2}-\frac{a'(t;p)}{2}+b(t;p)\Big]\\ = & \frac{1}{cb(t;p)}\Big[\Big(\frac{[b'(t;p)]^2}{16[b(t;p)]^2}-\frac{[a(t;p)]^2}{4}\Big)\\ & -\frac{b(t;p)b''(t;p)-[b'(t;p)]^2}{4[b(t;p)]^2}-\frac{a'(t;p)}{2}+b(t;p)\Big]\\ = & \frac{5}{16c}\frac{[b'(t;p)]^2}{(b(t;p))^3}-\frac{1}{4c}\frac{[a(t;p)]^2}{b(t;p)}\\ & -\frac{1}{4c}\frac{b''(t;p)}{[b(t;p)]^2}-\frac{a'(t;p)}{2cb(t;p)}+\frac{1}{c}.\end{aligned}$$

Therefore equation (3.2) becomes

$$y''(v;p)+cy(v;p)=C(t,y(v;p);p)y(v;p), \quad v\in J, \tag{3.10}$$

where

$$\begin{aligned}C(t,u;p):= & -\frac{5c}{16}\frac{[b'(t;p)]^2}{[b(t;p)]^3}+c\frac{[a(t;p)]^2}{4b(t;p)}\\ & +\frac{c}{4}\frac{b''(t;p)}{[b(t;p)]^2}+c\frac{a'(t;p)}{2b(t;p)}-c\frac{f(t,Y(t;p)u)}{b(t;p)Y(t;p)u}.\end{aligned}$$

(Recall that $c=\pm 1$, thus $c^2=1$.) The expression of the function $C(t,u;p)$ might assume a certain kind of singularity for $u=0$, but, as we shall see later, due to condition (3.13), such a case is impossible.

Therefore we have proved the *if* part of the following theorem:

Theorem 3.2. *Consider the differential equation* (1.5) *and assume that Condition 3.1 keeps in force. Then, a function* $y(v;p)$, $v\in J$ *is a solution of the differential equation* (3.10), *if and only if, the function*

$$x(t;p)=(S_p^{-1}y(\cdot;p))(t)=Y(t;p)y(v(t;p);p), \quad t\in[0,\hat{t}]$$

is a solution of (1.5). *The quantities* Y *and* v *are functions defined in* (3.7) *and* (3.4) *respectively.*

Proof. It is enough to prove the *only if* part. From the expression of $x(t;p)$ we get

$$x'(t;p)=Y'(t;p)y(v(t;p);p)+Y(t;p)v'(t;p)y'(v(t;p);p)$$

and

$$x''(t;p) = Y''(t;p)y(v(t;p);p) + 2Y'(t;p)v'(t;p)y'(v(t;p);p) \\ + Y(t;p)v''(t;p)y'(v(t;p);p) + Y(t;p)(v'(t;p))^2 y''(v(t;p);p).$$

Then, by using (3.5), (3.2) and the expression of the quantity $B(t;p)$, we obtain

$$x''(t) + a(t;p)x'(t) + b(t;p)x(t) + f(t,x(t);p) \\ = Y(t;p)(v'(t;p))^2\Big[y''(v(t;p);p) + B(t;p)y(v(t;p);p) + g(t;p)\Big] = 0.$$

□

To proceed we make the following condition:

Condition 3.3. *For each* $j = 1, 2, ..., 5$, *there is a nonnegative function* $\Phi_j \in \mathcal{A}_E$, *such that, for all* $t \in [0,T)$, $z \in \mathbb{R}$ *and large* p, *the inequalities*

$$|b'(t;p)|^2 \leq \Phi_1(p)|b(t;p)|^3, \\ |b''(t;p)| \leq \Phi_2(p)|b(t;p)|^2, \tag{3.11}$$

$$|a(t;p)|^2 \leq \Phi_3(p)|b(t;p)|, \\ |a'(t;p)| \leq \Phi_4(p)|b(t;p)|, \tag{3.12}$$

$$|f(t,z;p)| \leq \Phi_5(p)|zb(t;p)| \tag{3.13}$$

hold.

If Condition *3.3* is true, then we have the relation

$$\left|\frac{b'(0;p)}{b(0;p)}\right| \leq \sqrt{\Phi_1(p)b(0;p)}, \tag{3.14}$$

as well as the estimate

$$|C(t,u;p)| \leq \frac{5}{16}\Phi_1(p) + \frac{1}{4}\big(\Phi_2(p) + \Phi_3(p)\big) + \frac{1}{2}\Phi_4(p) + \Phi_5(p) \\ =: P(p), \tag{3.15}$$

for all $t \in [0,T)$ and p large enough.

4. Asymptotic approximation of the Initial Value Problem (1.5)-(1.6) in case $c = +1$

The previous facts will now help us to provide useful information on the asymptotic properties of the solutions of equation (1.5) having initial values which depend on the large parameter p, and are of the form (1.6).

In this subsection we assume that $c = +1$, thus the last requirement in Condition *3.1* keeps in force with $b(t;p) > 0$, for all $t \geq 0$ and p large enough.

As we have shown above, given a solution $x(t;p)$, $t \in [0, \hat{t}]$ of (1.5) the function $y(v;p)$, $v \in J$ defined in (3.1) solves equation (3.10) on the interval J. (Recall that J is the interval $[0, v(\hat{t};p)]$.) We shall find the images of the initial values (1.6) under this transformation.

First we note that

$$y(0;p) =: y_0(p) = \frac{x(0;p)}{Y(0;p)} = x(0;p) = x_0(p). \tag{4.1}$$

Also, from the fact that

$$x'(0;p) = Y'(0;p)y(0;p) + Y(0;p)v'(0;p)y'(0;p)$$

and relation (3.6) we obtain

$$y'(0;p) =: \hat{y}_0(p) = \frac{1}{\sqrt{b(0;p)}}\Big[\bar{x}_0(p) + \Big(\frac{b'(0;p)}{4b(0;p)} + \frac{a(0;p)}{2}\Big)x_0(p)\Big]. \tag{4.2}$$

Consider the solution $w(v;p)$ of the homogeneous equation

$$w'' + w = 0 \tag{4.3}$$

having the same initial values (4.1)-(4.2) as the function $y(\cdot;p)$. This requirement implies that the function $w(v;p)$ has the form

$$w(v;p) = c_1(p)\cos v + c_2(p)\sin v, \quad v \in \mathbb{R},$$

for some real numbers $c_1(p), c_2(p)$, which are uniquely determined by the initial values of $y(\cdot;p)$, namely $c_1(p) = y_0(p)$ and $c_2(p) = \hat{y}_0(p)$. Then the difference function

$$R(v;p) := y(v;p) - w(v;p), \tag{4.4}$$

satisfies

$$R(0;p) = R'(0;p) = 0,$$

and moreover

$$\begin{aligned} R''(v;p) + R(v;p) = {} & C(t, y(v;p);p)R(v;p) \\ & + C(t, y(v;p);p)w(v;p), \quad v \in J. \end{aligned} \tag{4.5}$$

Since the general solution of (4.3) having zero initial values is the zero function, applying the variation-of-constants formula in (4.5) we obtain

$$\begin{aligned} R(v;p) = {} & \int_0^v K(v,s)C(s;p;y(s;p))w(s;p)ds + \\ & + \int_0^v K(v,s)C(s;p;y(s;p))R(s;p)ds, \end{aligned} \tag{4.6}$$

where

$$K(v,s) = \sin(v-s).$$

Observe that

$$\int_0^v |\sin(v-s)w(s;p)|ds \leq (|c_1(p)|+|c_2(p)|)v =: \gamma(p)v, \quad v\in J$$

and therefore

$$|R(v;p)| \leq P(p)\gamma(p)v + P(p)\int_0^v |R(s;p)|ds.$$

Applying Gronwall's inequality we obtain

$$|R(v;p)| \leq \gamma(p)(e^{P(p)v}-1). \tag{4.7}$$

Differentiating $R(v;p)$ (with respect to v) in (4.6) and using (4.7), we see that the quantity $|R'(v;p)|$ has the same upper bound as $R(v;p)$ namely, we obtain

$$\max\{|R(v;p)|, |R'(v;p)|\} \leq \gamma(p)(e^{P(p)v}-1), \quad v\in J. \tag{4.8}$$

By using the transformation S_p and relation (4.8) we get the following theorem:

Theorem 4.1. *Consider the ordinary differential equation* (1.5) *associated with the initial values* (1.6), *where assume that* $T_0 = +\infty$ *and Condition 3.1 holds with* $c = +1$. *Assume also that there exist functions* Φ_j, $j = 1, 2, ..., 5$, *satisfying Condition 3.3. If* $x(t;p)$, $t \in [0,T)$ *is a maximally defined solution of the problem* (1.5)-(1.6), *then it holds*

$$T = +\infty, \tag{4.9}$$

and

$$\begin{aligned} |x(t;p)-&Y(t;p)w(v(t;p);p)| \\ &\leq \Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}} \exp\Big(-\frac{1}{2}\int_0^t a(s;p)ds\Big) \\ &\times \Big(\bar{x}_0(p) + \frac{1}{\sqrt{b(0;p)}}\big[|\bar{x}_0(p)| + |x_0(p)|\Big|\frac{b'(0;p)}{4b(0;p)} + \frac{a(0;p)}{2}\Big|\big]\Big) \\ &\times \Big[\exp\Big(P(p)\int_0^t \sqrt{b(s;p)}ds\Big) - 1\Big] =: \mathcal{M}(t;p), \end{aligned} \tag{4.10}$$

as well as

$$\Big|\frac{d}{dt}[x(t;p)-Y(t;p)w(v(t;p);p)]\Big|$$
$$\leq Y(t;p)\gamma(p)\big(e^{P(p)v(t;p)}-1\big)\Big[\frac{\sqrt{\Phi_1(p)b(t;p)}}{4}+\frac{|a(t;p)|}{2}+\sqrt{b(t;p)}\Big], \tag{4.11}$$

for all $t>0$ *and* p *large enough. Here we have set*

$$w(v;p):=x_0(p)cos(v)$$
$$+\frac{1}{\sqrt{b(0;p)}}\Big(\hat{x}_0(p)+\Big(\frac{b'(0;p)}{4b(0;p)}+\frac{a(0;p)}{2}\Big)x_0(p)\Big)\sin(v),$$

and $P(p)$ *is the quantity defined in* (3.15).

Proof. Inequality (4.10) is easily implied from (4.8) and the relation

$$x(t;p)=Y(t;p)y(v(t;p);p).$$

Then property (4.9) follows from (4.10) and the fact that the solution is noncontinuable (see, e.g., [16], p. 90).

To show (4.11) observe that

$$\Big|\frac{d}{dt}[x(t;p)-Y(t;p)w(v(t;p);p)]\Big|$$
$$=\Big|\frac{d}{dt}Y(t;p)[y(v(t;p);p)-w(v(t;p);p)]\Big|$$

and therefore

$$\Big|\frac{d}{dt}[x(t;p)-Y(t;p)w(v(t;p);p)]\Big|$$
$$\leq\Big|[y(v(t;p);p)-w(v(t;p);p)]\frac{d}{dt}Y(t;p)\Big|$$
$$+\Big|Y(t;p)\frac{d}{dt}[y(v(t;p);p)-w(v(t;p);p)]\Big||$$
$$\leq\Big|R(v(t;p);p)\frac{d}{dt}Y(t;p)\Big|+\Big|Y(t;p)\frac{d}{dv}R(v(t;p);p)\frac{d}{dt}v(t;p)\Big|$$
$$\leq Y(t;p)\gamma(p)\big(e^{P(p)v(t;p)}-1\big)\Big[\frac{\sqrt{\Phi_1(p)b(t;p)}}{4}+\frac{|a(t;p)|}{2}+\sqrt{b(t;p)}\Big].$$

We have used relations (3.8), (3.14) and (4.7). □

Now we present the main results concerning the existence of approximate solutions of the initial value problem (1.5) - (1.6).

The function defined by

$$\begin{aligned}\tilde{x}(t;p) : &= Y(t;p)w(v(t;p);p)\\ &= Y(t;p)[y_0(p)\cos\int_0^t \sqrt{b(s;p)}ds + \hat{y}_0(p)\sin\int_0^t \sqrt{b(s;p)}ds]\\ &= \Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}}\exp\Big(-\frac{1}{2}\int_0^t a(s;p)ds\Big)\Big\{x_0(p)\cos(v(t;p))\\ &+\frac{1}{\sqrt{b(0;p)}}\Big[\bar{x}_0(p)+\Big(\frac{b'(0;p)}{4b(0;p)}+\frac{a(0;p)}{2}\Big)x_0(p)\sin(v(t;p))\Big]\Big\}\end{aligned} \tag{4.12}$$

is the so called *approximate solution* of the problem, since, as we shall see in the sequel, this function approaches the exact solution as the parameter tends to $+\infty$. Moreover, since this function approaches the solution x in the C^1 sense, namely in a sense given in the next theorem, we shall refer to it as *a C^1 approximate solution.*

To make the notation short consider the *error* function

$$\mathcal{E}(t;p) := x(t;p) - \tilde{x}(t;p). \tag{4.13}$$

Then, from (4.10) and (4.11), we get

$$|\mathcal{E}(t;p)| \leq \mathcal{M}(t;p) \tag{4.14}$$

and

$$\begin{aligned}|\frac{d}{dt}\mathcal{E}(t;p)| &\leq Y(t;p)\gamma(p)\big(e^{P(p)v(t;p)}-1\big)\\ &\times\Big[\frac{\sqrt{\Phi_1(p)b(t;p)}}{4}+\frac{|a(t;p)|}{2}+\sqrt{b(t;p)}\Big],\end{aligned} \tag{4.15}$$

respectively.

Theorem 4.2. *Consider the initial value problem* (1.5) - (1.6)*, where the conditions of Theorem 4.1 keep in force and the relation*

$$\min_{j=1}^{5}\mathcal{G}_E(\Phi_j) > 0 \tag{4.16}$$

is satisfied. Moreover, we assume that

$$x_0,\ \ x_1 \in \mathcal{A}_E, \tag{4.17}$$

$$a(\cdot;p) \geq 0,\ \ \textit{for all large } p, \tag{4.18}$$

as well as

$$a(t;\cdot),\ \ b(t;\cdot) \in \mathcal{A}_E,\ \ t \in C_o(\mathbb{R}^+). \tag{4.19}$$

If $\mathcal{E}(t;p)$ *is the error function defined in* (4.13) *and the relation*

$$\begin{aligned}\min_{j=1}^{5}\mathcal{G}_E(\Phi_j)+\Big[\frac{3}{4}\mathcal{G}_E(b(t;\cdot)+\min\{\mathcal{G}_E(\bar{x}_0),\\ \mathcal{G}_E(x_0)+\frac{1}{2}\mathcal{G}_E(b(t;\cdot),\\ \mathcal{G}_E(x_0)+\mathcal{G}_E(a(t;\cdot)\}\Big]=:N_0>0,\ \ t\in C_o(\mathbb{R}),\end{aligned}\tag{4.20}$$

is satisfied, then we have

$$\mathcal{E}(t;p)\simeq 0,\ \ p\to+\infty,\ \ t\in C_o(\mathbb{R}^+)\tag{4.21}$$

and the growth index of the error function satisfies

$$\mathcal{G}_E(\mathcal{E}(t;\cdot))\geq N_0,\ \ t\in C_o(\mathbb{R}^+).\tag{4.22}$$

In addition to the assumptions above for the functions $x_0,\bar{x}_0,a,b$ *assume the condition*

$$\begin{aligned}\min_{j=1}^{5}\mathcal{G}_E(\Phi_j)+\Big[\frac{3}{4}\mathcal{G}_E(b(t;\cdot)+\min\{\mathcal{G}_E(\bar{x}_0)+\mathcal{G}_E(a(t;\cdot),\\ \frac{1}{2}\mathcal{G}_E(b(t;\cdot)+\mathcal{G}_E(\bar{x}_0),\mathcal{G}_E(x_0)+\mathcal{G}_E(b(t;\cdot),\\ \mathcal{G}_E(x_0)+2\mathcal{G}_E(a(t;\cdot),\frac{1}{2}\mathcal{G}_E(b(t;\cdot)+\mathcal{G}_E(x_0)\\ +\mathcal{G}_E(a(t;\cdot)\}\Big]=:N_1>0,\ \ t\in C_o(\mathbb{R}^+),\end{aligned}\tag{4.23}$$

instead of (4.20)*. Then we have*

$$\frac{d}{dt}\mathcal{E}(t;p)\simeq 0,\ \ p\to+\infty,\ \ t\in Co(\mathbb{R}^+),\tag{4.24}$$

and the growth index at infinity of the error function is such that

$$\mathcal{G}_E\big(\frac{d}{dt}\mathcal{E}(t;\cdot)\big)\geq N_1,\ \ t\in C_o(\mathbb{R}^+).\tag{4.25}$$

Proof. Due to our assumptions given $\varepsilon>0$ small enough, we can find real numbers σ,τ, and μ,ν, close to the quantities $-\mathcal{G}_E(x_0)$, $-\mathcal{G}_E(\bar{x}_0)$, $-\mathcal{G}_E(a(t;\cdot))$ and $-\mathcal{G}_E(b(t;\cdot))$ respectively, such that, as $p\to+\infty$,

$$x_0(p)=O(E(p))^{\sigma}),\ \ \bar{x}_0(p)=O((E(p))^{\tau}),\tag{4.26}$$

$$a(t;p)=O((E(p))^{\nu}),\quad \text{as } p\to+\infty,\ \ t\in C_o(\mathbb{R}^+)\tag{4.27}$$

and

$$b(t;p)=O((E(p))^{\mu}),\ \text{as } p\to+\infty,\ \ t\in C_o(\mathbb{R}^+),\tag{4.28}$$

as well as the relation

$$\min_{j=1}^{5}\mathcal{G}_E(\Phi_j)-\Big[\frac{3\mu}{4}+\max\{\tau,\sigma+\frac{\mu}{2},\sigma+\nu\}\Big]=:N_0-\varepsilon>0.\tag{4.29}$$

for

Assume that (4.18) holds. We start with the proof of (4.21). Fix any $\hat{t} > 0$ and take any $N \in (0, N_0 - \varepsilon)$. Then, due to (4.16), we can let $\zeta > 0$ such that

$$\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) > \zeta > N + \Big[\frac{3\mu}{4} + \max\{\tau, \sigma + \frac{\mu}{2}, \sigma + \nu\}\Big].$$

Therefore we have

$$\max\{\frac{3\mu}{4} + \tau,\ \frac{5\mu}{4} + \sigma,\ \frac{3\mu}{4} + \sigma + \nu\} - \zeta < -N, \tag{4.30}$$

and, due to Lemma 2.2, it holds

$$\mathcal{G}_E(P) > \zeta,\ \ \mathcal{G}_E(\Phi_1) > \zeta. \tag{4.31}$$

The latter implies that there exist $K > 0$ and $p_0 > 1$ such that

$$\begin{aligned} 0 &< P(p) \leq K(E(p))^{-\zeta}, \\ 0 &< \Phi_1(p) \leq K(E(p))^{-\zeta}, \end{aligned} \tag{4.32}$$

for all $p \geq p_0$.

From relations (4.27), (4.28) and (4.26) it follows that there are positive real numbers $K_j, j = 1, 2, 3, 4$ such that

$$|b(t; p) \leq K_1(E(p))^{\mu}, \tag{4.33}$$

$$\begin{aligned} |\bar{x}_0(p)| &\leq K_2(E(p))^{\tau}, \\ |x_0(p)| &\leq K_3(E(p))^{\sigma}, \end{aligned} \tag{4.34}$$

$$0 \leq a(t; p) \leq K_4(E(p))^{\nu}, \tag{4.35}$$

for all $t \geq 0$ and $p \geq p_1$, where $p_1 \geq p_0$.

Also keep in mind that from Condition 3.1 we have

$$b(t; p) \geq \theta, \tag{4.36}$$

In the sequel, for simplicity, we shall denote by q the quantity $E(p)$.

Consider the function $\mathcal{M}(t; p)$ defined in (4.10). Then, due to (4.32), (3.14) and (4.33)-(4.36), for all $t \in [0, \hat{t}]$ and p with $q = E(p) \geq p_1$, we have

$$\begin{aligned} \mathcal{M}(t; p) \leq K_1^{\frac{1}{4}} \theta^{-\frac{1}{4}} q^{\frac{\mu}{4}} \Big[K_2 q^{\tau} + \theta^{-\frac{1}{2}} \Big(K_2 q^{\tau} + K_3 q^{\sigma} \big[\frac{1}{4}(K_1 K)^{\frac{1}{2}} q^{\frac{-\zeta+\mu}{2}} \\ + \frac{1}{2} K_4 q^{\nu}\big]\Big)\Big] \Big(\sum_{n=1}^{+\infty} \frac{1}{n!} K^n q^{-n\zeta} t^n (K_1)^{\frac{n}{2}} q^{\frac{n\mu}{2}}\Big). \end{aligned} \tag{4.37}$$

Since the series

$$1+\sum_{n=1}^{+\infty}\frac{1}{(n+1)!}(tK)^n q^{-n\zeta}(K_1)^{\frac{n}{2}}q^{\frac{n\mu}{2}}$$

converges uniformly for t in compact sets, it admits an upper bound $K_5(\hat{t})>0$, say, on $[0,\hat{t}]$. Therefore, for all $t\in[0,\hat{t}]$ and $q=E(p)\geq p_1$, it holds

$$\sum_{n=1}^{+\infty}\frac{1}{n!}K^n q^{-n\zeta}t^n(K_1)^{\frac{n}{2}}q^{\frac{n\mu}{2}}\leq K_5(\hat{t})\hat{t}Kq^{-\zeta}(K_1)^{\frac{1}{2}}q^{\frac{\mu}{2}}.$$

So, from (4.30) and (4.37) we get

$$\begin{aligned}\mathcal{M}(t;p)&\leq K_1^{\frac{1}{4}}\theta^{-\frac{1}{4}}q^{\frac{\mu}{4}}\times\Big[(1+\theta^{-\frac{1}{2}})K_2q^{\tau}\\&\quad+K_3\theta^{-\frac{1}{2}}q^{\sigma}\frac{1}{4}(K_1K)^{\frac{1}{2}}q^{\frac{-\zeta+\mu}{2}}+K_3\theta^{-\frac{1}{2}}q^{\sigma}\frac{1}{2}K_4q^{\nu}\Big]\\&\quad\times K_5(\hat{t})\hat{t}Kq^{-\zeta}(K_1)^{\frac{1}{2}}q^{\frac{\mu}{2}}\\&=K_6q^{\frac{\mu}{4}+\tau-\zeta+\frac{\mu}{2}}+K_7q^{\frac{\mu}{4}+\sigma+\frac{-\zeta+\mu}{2}-\zeta+\frac{\mu}{2}}+K_8q^{\frac{\mu}{4}+\sigma+\nu-\zeta+\frac{\mu}{2}}\\&\leq K_6q^{-N}+K_7q^{-N-\frac{\zeta}{2}}+K_8q^{-N}<K_9q^{-N},\end{aligned}\tag{4.38}$$

for some positive constants $K_j, j=6,7,8,9$. Recall that

$$q=E(p)\geq p_1\geq p_0>1.$$

This and (4.14) complete the proof of (4.21).

Now, from the previous arguments it follows that given any $\Lambda\in(0,N)$ it holds

$$\mathcal{M}(t;p)q^{\Lambda}\leq K_9q^{-N+\Lambda}\to 0,\ \text{ as } p\to+\infty,$$

where the constant K_9 is uniformly chosen for t in the compact interval $[0,\hat{t}]$. Then from (4.14) we get

$$\mathcal{E}(t;p)q^{\Lambda}\to 0,\ \text{ as } p\to+\infty,$$

which implies that the growth index at infinity of the error function satisfies

$$\mathcal{G}_E(\mathcal{E}(t;p))\geq\Lambda.$$

From here we get

$$\mathcal{G}_E(\mathcal{E}(t;p))\geq N.$$

Since N is arbitrary in the interval $(0,N_0-\varepsilon)$ and ε is any small positive number, we obtain (4.22).

We proceed to the proof of (4.24).

Again, from our assumptions and (4.23), for any small enough $\varepsilon > 0$, we can choose real numbers σ, τ, and μ, ν, as above, satisfying (4.33), (4.34), (4.35), as well as

$$\begin{aligned}\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) - \Big[\frac{3\mu}{4} + \max\{\tau + \nu, \\ \frac{\mu}{2} + \tau, \sigma + \mu, \sigma + 2\nu, \frac{\mu}{2} + \sigma + \nu\}\Big] \\ =: N_1 - \varepsilon > 0.\end{aligned} \tag{4.39}$$

Take any $N \in (0, N_1 - \varepsilon)$. Then, because of (4.39), we can choose $\zeta > 0$ such that

$$\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) > \zeta > N + \Big[\frac{3\mu}{4} + \max\{\tau + \nu, \frac{\mu}{2} + \tau, \sigma + \mu, \sigma + 2\nu, \frac{\mu}{2} + \sigma + \nu\}\Big],$$

From this relation it follows that

$$\begin{aligned}\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) > & N + \Big[\frac{3\mu}{4} + \max\{\frac{\mu}{2} + \tau, \sigma + \mu, \frac{\mu}{2} + \sigma + \nu\}\Big] \\ &= (N + \frac{\mu}{2}) + \Big[\frac{3\mu}{4} + \max\{\tau, \sigma + \frac{\mu}{2}, \sigma + \nu\}\Big]\end{aligned}$$

and

$$\begin{aligned}\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) > & N + \Big[\frac{3\mu}{4} + \max\{\tau + \nu, \sigma + 2\nu, \frac{\mu}{2} + \sigma + \nu\}\Big] \\ &= (N + \nu) + \Big[\frac{3\mu}{4} + \max\{\tau, \sigma + \nu, \sigma + \frac{\mu}{2}\}\Big].\end{aligned}$$

These inequalities with a double use of (4.38), with N being replaced with

$$N + \frac{\mu}{2} \text{ and } N + \nu$$

respectively imply that

$$\mathcal{M}(t;p) < K_9 q^{-N-\frac{\mu}{2}} \text{ and } \mathcal{M}(t;p) < K_9 q^{-N-\nu}.$$

Then, from (4.15), (4.32) and conditions (4.33), (4.35) it follows that there are constants K_{10}, K_{11}, K_{12} such that

$$\begin{aligned}|\frac{d}{dt}\mathcal{E}(t;p)| \leq &\mathcal{M}(t;p)\Big[\frac{\sqrt{\Phi_1(p)b(t;p)}}{4}+\frac{|a(t;p)|}{2}+\sqrt{b(t;p)}\Big]\\ &\leq \mathcal{M}(t;p)[K_{10}q^{-\frac{\zeta}{2}}q^{\frac{\mu}{2}}+K_{11}q^{\nu}+K_{12}q^{\frac{\mu}{2}}]\\ &\leq K_{10}K_9q^{-N-\frac{\mu}{2}}q^{-\frac{\zeta}{2}}q^{\frac{\mu}{2}}+K_{11}K_9q^{-N-\nu}p^{\nu}\\ &\quad +K_{12}K_9q^{-N-\frac{\mu}{2}}q^{\frac{\mu}{2}}\\ &= K_{10}K_9q^{-N-\frac{\zeta}{2}}+K_{11}K_9q^{-N}+K_{12}K_9q^{-N}\\ &\leq (K_{10}+K_{11}+K_{12})q^{-N}.\end{aligned} \tag{4.40}$$

Since N is arbitrary, this relation completes the proof of (4.24).

Relation (4.25) follows from (4.40), exactly in the same way as (4.22) follows from (4.38). □

Theorem 4.3. *Consider the initial value problem* (1.5) - (1.6), *where the conditions of Theorem 4.1 and conditions* (4.17), (4.18), (4.19) *keep in force. Moreover assume that there is a measurable function* $\omega : [0,+\infty) \to [0,+\infty)$ *such that*

$$|a(t;p)| \leq \omega(t)\log(E(p)), \quad t \geq 0 \tag{4.41}$$

for p large enough. If $\mathcal{E}(t;p)$ *is the error function defined in* (4.13) *and the relation*

$$\begin{aligned}\min_{j=1}^{5}\lambda(\Phi_j)+\Big[&\frac{3}{4}\mathcal{G}_E(b(t;\cdot)+\min\{\mathcal{G}_E(\bar{x}_0),\mathcal{G}_E(x_0)\\ &+\frac{1}{2}\mathcal{G}_E(b(t;\cdot)),\mathcal{G}_E(x_0)\}\Big]\\ &=: M_0>0,\end{aligned} \tag{4.42}$$

holds, then we have

$$\mathcal{E}(t;p) \simeq 0, \quad p \to +\infty, \quad t \in Co([0,T(M_0))), \tag{4.43}$$

where, for any $M>0$ *we have set*

$$T(M) := \sup\{t>0 : \Omega(t) := \int_0^t \omega(s)ds < 2M\}. \tag{4.44}$$

In this case the growth index of the error function satisfies

$$\mathcal{G}_E(\mathcal{E}(t;\cdot)) \geq M_0, \quad t \in Co([0,T(M_0))). \tag{4.45}$$

Also, if (4.41) *keeps in force and the condition*

$$\min_{j=1}^{5}\lambda(\Phi_j)+\Big[\frac{3}{4}\mathcal{G}_E(b(t;\cdot))+\min\{\mathcal{G}_E(\bar{x}_0),\frac{1}{2}\mathcal{G}_E(b(t;\cdot))+\mathcal{G}_E(\bar{x}_0),\mathcal{G}_E(x_0)+\mathcal{G}_E(b(t;\cdot)),\mathcal{G}_E(x_0),\frac{1}{2}\mathcal{G}_E(b(t;\cdot))+\mathcal{G}_E(x_0)\}\Big]=:M_1>0 \tag{4.46}$$

is satisfied, then we have

$$\frac{d}{dt}\mathcal{E}(t;p)\simeq 0,\ \ p\to+\infty,\ \ t\in Co([0,T(M_1))) \tag{4.47}$$

and the growth index of the error function is such that

$$\mathcal{G}_E(\frac{d}{dt}\mathcal{E}(t;\cdot))\geq M_1,\ \ t\in Co([0,T(M_1))). \tag{4.48}$$

Proof. Let $\hat{t}\in(0,T(M_0))$ be fixed. Then from (4.42) we can choose numbers μ,σ,τ satisfying (4.33) and (4.34) and such that $-\mu,-\sigma,-\tau$ are close to $\mathcal{G}_E(b(t;\cdot))$, $\mathcal{G}_E(x_0)$ and$\mathcal{G}_E(\bar{x}_0)$, respectively and moreover

$$\Big[\frac{3\mu}{4}+\max\{\tau,\sigma+\frac{\mu}{2},\sigma\}\Big]+\frac{1}{2}\Omega(\hat{t})<\min_{j=1}^{5}\mathcal{G}_E(\Phi_j).$$

Take ζ,ν,N (strictly) positive such that

$$\Big[\frac{3\mu}{4}+\max\{\tau,\sigma+\frac{\mu}{2},\sigma+\nu\}\Big]+\frac{1}{2}\Omega(\hat{t})+N<\zeta<\min_{j=1}^{5}\mathcal{G}_E(\Phi_j). \tag{4.49}$$

Let $p_0>1$ be chosen so that $\log(p)\leq p^{\nu}$ and (4.41) holds, for all $p\geq p_0$. Then, due to (4.41), we have

$$|a(0;p)|\leq\omega(0)q^{\nu}, \tag{4.50}$$

for all $p\geq p_0$. Recall that $q:=E(p)$.

Now we proceed as in Theorem 4.2, where, due to (4.41) and (4.50), relation (4.38) becomes

$$\begin{aligned}\mathcal{M}(t;p)&\leq K_1^{\frac{1}{4}}\theta^{-\frac{1}{4}}q^{\frac{\mu}{4}}e^{\frac{1}{2}\Omega(\hat{t})\log(q)}\\&\times\Big[(1+\theta^{-\frac{1}{2}})K_2q^{\tau}+K_3\theta^{-\frac{1}{2}}q^{\sigma}\frac{1}{4}(K_1K)^{\frac{1}{2}}q^{\frac{-\zeta+\mu}{2}}\\&\quad+K_3\theta^{-\frac{1}{2}}q^{\sigma}\frac{1}{2}\omega(0)\log(q)\Big]\times K_5(\hat{t})\hat{t}Kq^{-\zeta}(K_1)^{\frac{1}{2}}q^{\frac{\mu}{2}}\\&\leq K_6q^{\frac{\mu}{4}+\tau-\zeta+\frac{\mu}{2}+\frac{1}{2}\Omega(\hat{t})}\\&\quad+K_7q^{\frac{\mu}{4}+\sigma+\frac{-\zeta+\mu}{2}-\zeta+\frac{\mu}{2}+\frac{1}{2}\Omega(\hat{t})}+K_8q^{\frac{\mu}{4}+\sigma+\nu-\zeta+\frac{\mu}{2}+\frac{1}{2}\Omega(\hat{t})}.\end{aligned} \tag{4.51}$$

Notice that (4.51) holds for all $q := E(p)$ with $p \geq p_0 > 1$. From this inequality and (4.49) we obtain the estimate

$$\mathcal{M}(t;p) \leq (K_6 + K_7 + K_8)q^{-N}, \tag{4.52}$$

which implies the approximation (4.43). Inequality (4.45) follows as the corresponding one in Theorem 4.2. Finally, as in Theorem 4.2, we can use the above procedure and (4.52) in order to get a relation similar to (4.40), from which (4.47) and (4.48) follow. □

5. Application to the Initial Value Problem (1.3)-(1.4)

Consider the initial value problem (1.3)-(1.4), where assume the following conditions:

(i) The function $b_1 \in C^2([0,+\infty),[0,+\infty))$ it is bounded and it has bounded derivatives.

(ii) The functions $a_1, a_2 \in C^1([0,+\infty),[0,+\infty))$ are bounded with bounded derivatives.

(iii) The function b_2 is a nonzero positive constant and, as we said previously, the exponents $\mu, \nu, m, \sigma, \tau$ of the model are real numbers.

Observe that Condition 3.3 is satisfied by choosing the following functions:

$$\Phi_1(p) = l_1 p^{-3\mu}, \quad \Phi_2(p) = l_2 p^{-2\mu}, \quad \Phi_3(p) = l_3 p^{2\nu-\mu},$$

$$\Phi_4(p) = l_4 p^{\nu-\mu}, \quad \Phi_5(p) = l_5 p^{m-\mu},$$

for some positive constants l_j, $j = 1, 2, \cdots, 5$. It is not hard to show that the growth index of these functions with respect to the function $E(p) := p$, are

$$\mathcal{G}_E(\Phi_1) = 3\mu, \quad \mathcal{G}_E(\Phi_2) = 2\mu, \quad \mathcal{G}_E(\Phi_3) = -2\nu + \mu,$$

$$\mathcal{G}_E(\Phi_4) = -\nu + \mu, \quad \mathcal{G}_E(\Phi_5) = -m + \mu.$$

In this case the results (4.21) - (4.22) and (4.24) - (4.25) keep in force with N_0 and N_1 being defined as

$$N_0 := \min\{\frac{5\mu}{4}, \frac{\mu}{4} - 2\nu, \frac{\mu}{4} - m\} - \max\{\tau, \frac{\mu}{2} + \sigma, \sigma + \nu\}$$

and

$$N_1 = \min\{\frac{5\mu}{4}, \frac{\mu}{4} - 2\nu, \frac{\mu}{4} - m\} - \max\{\tau+\nu, \mu+\sigma, \frac{\mu}{2}+\tau, \sigma+2\nu, \frac{\mu}{2}+\sigma+\nu\},$$

respectively, provided that they are positive.

To give a specific application let us assume that the functions a_1, a_2, b_1 are constants. Then we can obtain the approximate solution of the initial value problem (1.3)-(1.4) by finding the error function.

Indeed, via (4.12), we can see that a C^1-approximate solution of problem (1.3)-(1.4) is the function defined by

$$\begin{aligned}\tilde{x}(t;p) :=& exp[-\frac{1}{2}t(a_1 + a_2p^\nu)]\\ &\times \Big[(\delta_1 + \delta_2 p^\sigma)\cos[t(b_1 + b_2p^\mu)] + (b_1 + b_2p^\mu)^{-\frac{1}{2}}\\ &\times \Big(\eta_1 + \eta_2 p^\tau + \frac{1}{2}(\delta_1 + \delta_2 p^\sigma)(a_1 + a_2 p^\nu)\Big)\\ &\times \sin[t(b_1 + b_2 p^\mu)]\Big], \quad t \geq 0.\end{aligned}$$

This approximation is uniform for t in compact intervals of the positive real axis. For instance, for the values

$$\begin{aligned}&a_1 = 2, \ \ a_2 = \delta_2 = 0, \ \ \delta_1 = b_1 = b_2 = \eta_1 = \eta_2 = 1\\ &\mu = \frac{9}{10}, \ \ \nu = \frac{1}{10}, \ \ m < 0, \ \ \tau = -\frac{9}{20}, \sigma = -1,\end{aligned} \tag{5.1}$$

we can find that the growth index at infinity of the error function $\mathcal{E}(t;\cdot))$ satisfies

$$\mathcal{G}_E(\mathcal{E}(t;\cdot)) \geq \frac{19}{40} \text{ and } \mathcal{G}_E(\frac{d}{dt}\mathcal{E}(t;\cdot)) \geq \frac{1}{40}.$$

In Figure 1 the approximate solution for the values p=50, $p = 150$ and p=250 are shown.

6. Approximate solutions of the Initial Value Problem (1.5)-(1.6) in case $c = -1$

In this section we shall discuss the IVP (1.5)-(1.6) when $c = -1$, thus we assume that $b(t;p) < 0$, for all t and large p. We shall assume throughout of this section that Condition 3.3 (given in the end of Section 3) is satisfied.

Here the function y defined in (3.1) takes initial values $y_0(p)$ and $\hat{y}_0(p)$ as in (4.1) and (4.2). We wish to proceed as in Section 4 and consider a fixed point $\hat{t} > 0$, as well as the solution

$$w(v;p) := c_1(p)e^v + c_2(p)e^{-v}, \ \ v \in [0, \hat{v}]$$

of equation

$$w'' - w = 0, \tag{6.1}$$

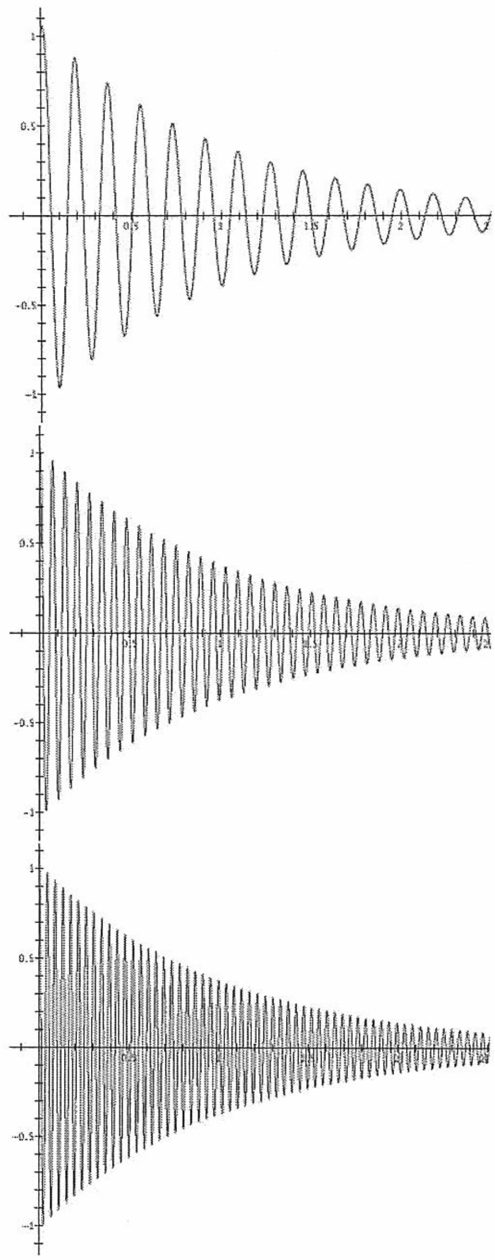

FIGURE 1. Approximate solutions of the problem (1.3)-(1.4), with the values (5.1) and when $p = 50$, $p = 150$ and $p = 250$, respectively

associated with the same initial values as y. We have set $\hat{v} := v(\hat{t};p)$. Thus, for $j=1,2$ we obtain

$$c_j(p) = \frac{1}{2}\Big[x_0(p) - \frac{(-1)^j}{\sqrt{-b(0;p)}}\Big(\bar{x}_0(p) + x_0(p)\big[\frac{b'(0;p)}{4b(0;p)} + \frac{a(0;p)}{2}\big]\Big)\Big]$$

and therefore it holds

$$\begin{aligned}|c_j(p)| \le \frac{1}{2}\Big[|x_0(p)| + \frac{1}{\sqrt{-b(0;p)}}\Big|\bar{x}_0(p)\\ + x_0(p)\big[\frac{b'(0;p)}{4b(0;p)} + \frac{a(0;p)}{2}\big]\Big|\Big] =: \kappa(p).\end{aligned} \tag{6.2}$$

Also, the difference function R defined in (4.4) satisfies (4.6) where, now, we have

$$K(v,s) = \sinh(v-s).$$

Observe that

$$\begin{aligned}\int_0^v \sinh(v-s)|w(s;p)|ds &\le \frac{|c_1(p)|}{2}\int_0^v (e^{v-s} - e^{-v+s})e^s ds\\ &+ \frac{|c_2(p)|}{2}\int_0^v (e^{v-s} - e^{-v+s})e^{-s} ds\\ &\le \frac{|c_1(p)|}{2}(ve^v - \sinh(v))\\ &+ \frac{|c_2(p)|}{2}(\sinh(v) - ve^{-v})\\ &\le \kappa(p)v\sinh(v)\end{aligned}$$

and therefore, for any $v \in [0,\hat{v}]$, it holds

$$\begin{aligned}|R(v;p)| &\le P(p)\kappa(p)v\sinh(v) + P(p)\int_0^v \sinh(v-s)|R(s;p)|ds\\ &\le P(p)\kappa(p)v\sinh(v) + P(p)\sinh(v)\int_0^v |R(s;p)|ds.\end{aligned}$$

Here we apply the method of proving Gronwall's inequality, but we follow a different procedure. Indeed, we set

$$F(v) := \int_0^v |R(s;p)|ds.$$

Then

$$F'(v) = |R(v;p)| \le P(p)\kappa(p)v\sinh(v) + P(p)\sinh(v)F(v)$$

and therefore

$$F'(v) - P(p)\sinh(v)F(v) \le P(p)\kappa(p)v\sinh(v).$$

Multiply both sides with the factor exp $\Big(-P(p)\cosh(v)\Big)$ and integrate from 0to v. Then we obtain

$$\begin{aligned}F(v)e^{-P(p)\cosh(v)} &\leq P(p)\kappa(p)\int_0^v s\sinh(s)e^{-P(p)\cosh(s)}ds\\ &= \kappa(p)(-ve^{-P(p)\cosh(v)} + \int_0^v e^{-P(p)\cosh(s)}ds\\ &\leq \kappa(p)v(1-e^{-P(p)\cosh(v)}).\end{aligned}$$

Therefore we have

$$\begin{aligned}|R(v;p)| &\leq P(p)\kappa(p)v\sinh(v)\\ &+ P(p)\kappa(v)v\sinh(v)(e^{P(p)\cosh(v)}-1)\\ &= P(p)\kappa(p)v\sinh(v)e^{P(p)\cosh(v)}.\end{aligned} \tag{6.3}$$

Next we observe that

$$\begin{aligned}\int_0^v \cosh(v-s)|w(s;p)|ds &\leq \frac{|c_1(p)|}{2}(ve^v + \sinh(v))\\ &+ \frac{|c_2(p)|}{2}(\sinh(v) + ve^{-v})\\ &\leq \kappa(p)(v\cosh(v) + \sinh(v))\end{aligned}$$

and therefore, for any $v \in [0,\hat{v}]$, it holds

$$\begin{aligned}|R'(v;p)| &\leq P(p)\kappa(p)(v\cosh(v) + \sinh(v))\\ &+ P(p)\int_0^v \cosh(v-s)|R(s;p)|ds\\ &\leq P(p)\kappa(p)(v\cosh(v)\\ &+ \sinh(v)) + P(p)\cosh(v)\int_0^v |R(s;p)|ds.\end{aligned}$$

Using this inequality and (6.3) we obtain

$$\begin{aligned}|R'(v;p)| &\leq P(p)\kappa(p)(v\cosh(v) + \sinh(v))\\ &+ P(p)\kappa(p)e^{P(p)}v\cosh(v)(e^{P(p)(\cosh(v)-1)} - 1).\end{aligned} \tag{6.4}$$

The proof of the next theorem follows as the proof of Theorem 4.1, by using (6.3), (6.4) and the expression of the functions v and Y from (3.4) and (3.7) respectively. So we omit it.

Theorem 6.1. *Consider the ordinary differential equation*(1.5) *associated with the initial values* (1.6), *where assume that the Condition 3.1 holds with c = -1. Assume also that there exist functions* Φ_j,

$j = 1, 2, ..., 5$, *satisfying* (3.11), (3.12), (3.13). *If* $x(t;p)$, $t \in [0,T)$ *is a maximally defined solution of the problem* (1.5)-(1.6), *then it holds*

$$T = +\infty,$$

and if we set

$$w(v;p) := \frac{1}{2}\sum_{j=1}^{2} e^{-(-1)^j v}\Big[x_0(p) - \frac{(-1)^j}{\sqrt{-b(0;p)}}\Big(\bar{x}_0(p) + x_0(p)\big[\frac{b'(0;p)}{4b(0;p)} + \frac{a(0;p)}{2}\big]\Big)\Big]$$

and

$$\mathcal{E}(t;p) := x(t;p) - Y(t;p)w(v(t;p);p),$$

then we have

$$|\mathcal{E}(t;p)| \leq P(p)\kappa(p)\Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}} \exp\big(-\frac{1}{2}\int_0^t a(s;p)ds\big) \times \int_0^t \sqrt{-b(s;p)}ds \sinh\Big[\int_0^t \sqrt{-b(s;p)}ds\Big] \times \exp\big(P(p)\cosh(\int_0^t \sqrt{-b(s;p)}ds)\big) =: \mathcal{L}(t;p), \tag{6.5}$$

as well as

$$\Big|\frac{d}{dt}\mathcal{E}(t;p)\Big| \leq \mathcal{L}(t;p)\Big[\frac{\sqrt{\Phi_1(p)|b(t;p)|}}{4} + \frac{|a(t;p)|}{2}\Big] + \Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}} \exp\big(-\frac{1}{2}\int_0^t a(s;p)ds\big)\sqrt{-b(t;p)} \times P(p)\kappa(p)\Big[\big(\int_0^t \sqrt{-b(s;p)}ds\big)\cosh\big(\int_0^t \sqrt{-b(s;p)}ds\big) + \sinh\big(\int_0^t \sqrt{-b(s;p)}ds\big) + e^{P(p)}\int_0^t \sqrt{-b(s;p)}ds \cosh\big(\int_0^t \sqrt{-b(s;p)}ds\big) \times \Big(e^{P(p)(\cosh(\int_0^t \sqrt{-b(s;p)}ds)-1} - 1\Big)\Big], \tag{6.6}$$

for all $t \in I$ *and* p. *Here* P *is defined in* (3.15) *and* κ *in* (6.2).

Now we give the main results of this section.

Theorem 6.2. *Consider the initial value problem* (1.5) - (1.6), *where the conditions of Theorem 6.1 keep in force. Moreover assume that* $a(\cdot;p) \geq 0$, *for all large* p, *as well as the following properties:*

i) It holds $\sup_{t>0} b(t;p) < 0$, *uniformly for all* t *in compact sets and all large* p.

ii) It holds $\lambda(\Phi_j) > 0$, *for all* $j = 1, 2, ..., 5$.

iii) It holds $x_0, x_1 \in \mathcal{A}_E$.

Define the function

$$\tilde{x}(t;p) := \Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}} \exp\big(-\frac{1}{2}\int_0^t a(s;p)ds\big)$$
$$\times \frac{1}{2}\sum_{j=1}^{2} e^{-(-1)^j v}\Big[x_0(p) \tag{6.7}$$
$$- \frac{(-1)^j}{\sqrt{-b(0;p)}}\Big(\bar{x}_0(p) + x_0(p)\big[\frac{b'(0;p)}{4b(0;p)} + \frac{a(0;p)}{2}\big]\Big)\Big].$$

Let x *be a solution of the problem and we let* $\mathcal{E}(t;p)$ *be the error function defined by*

$$\mathcal{E}(t;p) := x(t;p) - \tilde{x}(t;p).$$

a) If $a(t;\cdot) \in \mathcal{A}_E$, $t \in Co(\mathbb{R}^+)$ *holds and there is a measurable function* $z(t)$, $t \geq 0$ *such that*

$$|b(t;p)| \leq z(t)\big[\log(\log(E(p)))\big]^2, \tag{6.8}$$

for all $t \geq 0$ *and* p *large enough, then we have*

$$\mathcal{E}(t;p) \simeq 0, \quad p \to +\infty, \quad t \in Co(\mathbb{R}^+), \tag{6.9}$$

provided that the quantities above satisfy the relation

$$\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) + \min\{\mathcal{G}_E(\bar{x}_0), \mathcal{G}_E(x_0), \mathcal{G}_E(x_0) + \mathcal{G}_E(a(t;\cdot))\} =: L_0 > 0, \quad t \in Co(\mathbb{R}^+). \tag{6.10}$$

The growth index of the error function satisfies

$$\mathcal{G}_E(\mathcal{E}(t;\cdot)) \geq L_0, \quad t \in Co(\mathbb{R}^+). \tag{6.11}$$

b) Assume that (6.8) *holds and* $z(t), t \geq 0$ *is a constant,* $z(t) = \eta$, *say. If the condition*

$$\min_{j=1}^{5}\mathcal{G}_E(\Phi_j) - 1 + \min\{\mathcal{G}_E(\bar{x}_0),\ \mathcal{G}_E(x_0),\ \mathcal{G}_E(x_0) + \mathcal{G}_E(a(t;\cdot)),\ \mathcal{G}_E(a(t;\cdot)) + \mathcal{G}_E(\bar{x}_0),\ \mathcal{G}_E(x_0) + 2\mathcal{G}_E(a(t;\cdot))\} =: L_1 > 0, \quad t \in Co(\mathbb{R}^+) \tag{6.12}$$

holds, then we have

$$\frac{d}{dt}\mathcal{E}(t;p) \simeq 0, \quad p \to +\infty, \quad t \in Co(\mathbb{R}^+), \tag{6.13}$$

and

$$\mathcal{G}_E\big(\frac{d}{dt}\mathcal{E}(t;\cdot)\big) \geq L_1, \quad t \in Co(\mathbb{R}^+). \tag{6.14}$$

Proof. We start with the proof of (6.9). Due to (6.10), given any small $\varepsilon > 0$ and $N \in (0, L_0 - \varepsilon)$ we take reals $\zeta > 0$ and τ, σ, ν near to $-\mathcal{G}_E(\hat{x}_0)$, $-\mathcal{G}_E(x_0)$, $-\mathcal{G}_E(a(t;\cdot))$ respectively, such that

$$\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) > \zeta > N + \max\{\tau, \sigma, \sigma + \nu\}.$$

Hence (4.34) and (4.35) keep in force. These arguments and Lemma 2.2 imply that (4.32) hold, for some $K > 0$ and $q := E(p)$ with $p \geq p_0 \geq 1$. Notice, also, that

$$\max\{\tau, \ \sigma, \ \sigma + \nu\} - \zeta < -N. \tag{6.15}$$

Because of (6.15) we can obtain some $\delta > 0$ and $p_1 \geq p_0$ such that

$$\frac{5\delta}{2} + \frac{K}{2}q^{-\zeta} + \max\{\tau, \ \sigma + \delta, \ \sigma + \nu\} - \zeta < -N, \ \ p \geq p_1. \tag{6.16}$$

Keep in mind assumption (i) of the theorem, relations (4.34) and (4.35), for some positive constants K_2, K_3, K_4 and, moreover,

$$b(t; p) \leq -\theta, \tag{6.17}$$

for all t and p large. Fix any $\hat{t} > 0$ and define

$$\lambda := \int_0^{\hat{t}} \sqrt{z(s)ds}.$$

Obviously there is a $p_2 \geq p_1$ such that for all $q \geq p_2$, we have

$$Kq^{-\zeta} \leq 1, \ \ q \geq p_2 \tag{6.18}$$

and

$$\log(\log(u)) \leq \log(u) \leq u^{\delta}, \ \ u \geq p_2. \tag{6.19}$$

Consider the function $\mathcal{L}(t; p)$ defined in (6.5). Then due to (4.32), (3.14), (4.33), (4.34), (6.17), (6.8) and (6.19), for all $t \in [0, \hat{t}]$ and

$q \geq p_2$, we have

$$\begin{aligned}\mathcal{L}(t;p) &\leq P(p)\kappa(p)\Big(\frac{z(0)}{\theta}\Big)^{\frac{1}{4}}\mathcal{G}_E\Big[\log(\log(q))\Big]^{\frac{3}{2}}\sinh[\mathcal{G}_E\log(\log(q))]\\ &\quad\times\exp\big[P(p)\cosh[\log(\log(q))]\big]\\ &\leq \mathcal{G}_E P(p)\kappa(p)\Big(\frac{z(0)}{\theta}\Big)^{\frac{1}{4}}q^{\frac{3\delta}{2}}\frac{1}{2}q^{\delta}p^{\frac{P(p)}{2}}\exp(\frac{\mathcal{G}_E P(p)}{2\log(q)})\\ &\leq \mathcal{G}_E K q^{-\zeta}\frac{1}{2}\Big[K_3q^{\sigma}+\frac{K_2}{\sqrt{\theta}}q^{\tau}+\frac{K_3}{\sqrt{\theta}}q^{\sigma}\Big(\frac{\sqrt{K}}{4}q^{-\frac{\zeta}{2}}q^{\delta}\sqrt{z(0)}+\frac{K_4}{2}q^{\nu}\Big)\Big]\\ &\quad\times\Big(\frac{z(0)}{\theta}\Big)^{\frac{1}{4}}q^{\frac{3\delta}{2}}\frac{1}{2}q^{\delta}q^{\frac{P(p)}{2}}\exp(\frac{\mathcal{G}_E P(p)}{2\log(p)})e^{\frac{1}{\lambda}}.\end{aligned}$$

Therefore it follows that

$$\begin{aligned}\mathcal{L}(t;p) &\leq \Lambda_1 q^{-\zeta+\sigma+\frac{3\delta}{2}+\delta+\frac{K}{2}q^{-\zeta}}+\Lambda_2 q^{-\zeta+\tau+\frac{3\delta}{2}+\delta+\frac{K}{2}q^{-\zeta}}\\ &\quad+\Lambda_3 q^{-\zeta+\sigma-\frac{\zeta}{2}+\delta+\frac{3\delta}{2}+\delta+\frac{K}{2}q^{-\zeta}}+\Lambda_4 q^{-\zeta+\sigma+\nu+\frac{3\delta}{2}+\delta+\frac{K}{2}q^{-\zeta}},\end{aligned} \tag{6.20}$$

for some constants $\Lambda_j, j = 1, 2, 3, 4$. From (6.16) and (6.20) we obtain

$$\mathcal{L}(t;p) \leq \Lambda_0 q^{-N}, \quad t \in [0, \hat{t}] \tag{6.21}$$

for some $\Lambda_0 > 0$. This and (6.5) complete the proof of (6.9).

Now, from the previous arguments it follows that given any $L \in (0, N)$ it holds

$$\mathcal{L}(t;p)q^L \leq \Lambda_0 q^{-N+L} \to 0, \text{ as } p \to +\infty,$$

where, notice that, the constant Λ_0 is uniformly chosen for t in the interval $[0, \hat{t}]$ and p with $E(p) \geq p_2$. This gives

$$\mathcal{E}(t;p)q^L \to 0, \text{ as } p \to +\infty, \ t \in Co(\mathbb{R}^+).$$

Hence the growth index of the error function $\mathcal{E}$ satisfies $\mathcal{G}_E(\mathcal{E}(t;p)) \geq L$ and so we get

$$\mathcal{G}_E(\mathcal{E}(t;p)) \geq N \text{ as } p \to +\infty.$$

Since N is arbitrary in the interval $(0, N_0 - \varepsilon)$ and ε is small, we get (6.11).

(b) Fix any $\hat{t} > 0$ and take any small $\varepsilon > 0$ and $N \in (0, L_1 - \varepsilon)$. Also from (6.12) we can get $\zeta > 0$, $\delta > 0$ and reals σ, ν, τ as above, such that

$$\begin{aligned}\max\Big\{&\frac{5\delta}{2}+1\\&+\max\{\delta+\sigma+\nu,\ \ \delta+\tau,\nu+\tau,\ \ 2\delta+\sigma,\ \ 2\nu+\sigma\},\\&2\delta+\hat{t}\sqrt{\eta}\delta+1+\max\{\tau,\delta+\sigma,\sigma+\nu\}\Big\}+N\\&<\zeta<\min_{j=1}^{5}\mathcal{G}_E(\Phi_j).\end{aligned}\tag{6.22}$$

Such a δ may be chosen in such way that

$$\hat{t}\sqrt{\eta}\delta<1.$$

By using inequality (6.6) and relation (3.8) we get

$$\begin{aligned}|\frac{d}{dt}\mathcal{E}(t;p)|\leq\ &\mathcal{L}(t;p)\Big[\frac{\sqrt{\Phi_1(p)|b(t;p)|}}{4}+\frac{|a(t;p)|}{2}\Big]\\&+\Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}}exp\Big(-\frac{1}{2}\int_0^{\hat{t}}a(s)ds\Big)\sqrt{-b(t;p)}\\&\times\Big[\int_0^t\sqrt{-b(s;p)}ds\cosh\Big(\int_0^t\sqrt{-b(s;p)}ds\Big)\\&\quad+\sinh\Big(\int_0^t\sqrt{-b(s;p)}ds\Big)\\&+e^{P(p)}\int_0^t\sqrt{-b(s;p)}ds\cosh\Big(\int_0^t\sqrt{-b(s;p)}ds\Big)\\&\times\Big(exp\Big(P(p)(\cosh\Big(\int_0^t\sqrt{-b(s;p)}ds\Big)-1)\Big)-1\Big)\Big],\end{aligned}$$

namely

$$\begin{aligned}|\frac{d}{dt}\mathcal{E}(t;p)|\leq\ &\mathcal{L}(t;p)\Big[\frac{1}{4}K^{\frac{1}{2}}p^{\frac{-\zeta}{2}}\sqrt{\eta}\log(\log(q))+\frac{K_4q^{\nu}}{2}\Big]\\&+\Big(\frac{\eta}{\theta}\Big)^{\frac{1}{4}}\big(\log(\log(q))\big)^{\frac{1}{2}}Kq^{-\zeta}\\&\times\frac{1}{2}\Big[K_3q^{\sigma}+\frac{1}{\sqrt{\eta}}\Big(K_2q^{\tau}\\&\quad+K_3q^{\sigma}\Big[\frac{1}{4}K^{\frac{1}{2}}q^{\frac{-\zeta}{2}}\sqrt{\eta}\log(\log(q))+\frac{K_4p^{\nu}}{2}\Big]\end{aligned}$$

$$\times\Big[\hat{t}\sqrt{\eta}(\log(\log(q)))\cosh(\hat{t}\sqrt{\eta}\log(\log(q))$$
$$+\sinh(\hat{t}\sqrt{\eta}\log(\log(q)))$$
$$+e^{Kq^{-\zeta}}\hat{t}\sqrt{\eta}\log(\log(q))\cosh(\hat{t}\sqrt{\eta}\log(\log(q)))$$
$$\times\Big(\exp\Big(Kq^{-\zeta}(\cosh(\hat{t}\sqrt{\eta}\log(\log(q)))-1)\Big)-1\Big)\Big]\lambda q^{\frac{\delta}{2}}.$$

Letting any p with $q:=E(p)\geq p_0>e$, and p_0 being such that

$$q\geq p_0\Longrightarrow \log(q)\leq q^{\delta}$$

and using the fact that

$$x>0\Longrightarrow \cosh(x)\leq e^{x}\ \text{ and }\ \sinh(x)\leq\frac{1}{2}e^{x},$$

from the previous estimate, we get

$$|\frac{d}{dt}\mathcal{E}(t;p)|\leq\Big[\Lambda_1 q^{-\zeta+\sigma+\frac{3\delta}{2}+\delta+\frac{K}{2}q^{-\zeta}}+\Lambda_2 q^{-\zeta+\tau+\frac{3\delta}{2}+\delta+\frac{K}{2}q^{-\zeta}}$$
$$+\Lambda_3 q^{-\zeta+\sigma-\frac{\zeta}{2}+\delta+\frac{3\delta}{2}+\delta+\frac{K}{2}q^{-\zeta}}+\Lambda_4 q^{-\zeta+\sigma+\nu+\frac{3\delta}{2}+\delta+\frac{K}{2}q^{-\zeta}}\Big]$$
$$\times\Big[\frac{1}{4}K^{\frac{1}{2}}q^{\frac{-\zeta}{2}}\sqrt{\eta}q^{\delta}+\frac{K_4q^{\nu}}{2}\Big]+\Big(\frac{\eta}{\theta}\Big)^{\frac{1}{4}}q^{\frac{\delta}{2}}Kq^{-\zeta}$$
$$\times\frac{1}{2}\Big[K_3q^{\sigma}+\frac{1}{\sqrt{\eta}}\Big(K_2q^{\tau}+K_3q^{\sigma}\Big[\frac{1}{4}K^{\frac{1}{2}}q^{\frac{-\zeta}{2}}\sqrt{\eta}q^{\delta}+\frac{K_4q^{\nu}}{2}\Big]$$
$$\times\Big[\hat{t}\sqrt{\eta}q^{\delta}p^{\hat{t}\sqrt{\eta}\delta}+\frac{1}{2}q^{\hat{t}\sqrt{\eta}\delta}+e^{Kq^{-\zeta}}\hat{t}\sqrt{\eta}q^{\delta}q^{\hat{t}\sqrt{\eta}\delta}$$
$$\times\Big(\exp\Big(Kq^{-\zeta}((\log(q))^{\hat{t}\sqrt{\eta}\delta})\Big)\Big]\lambda q^{\frac{\delta}{2}}.$$

Therefore it follows that

$$|\frac{d}{dt}\mathcal{E}(t;p)|\leq\sum_{j=1}^{20}\Gamma_j q^{r_j},\tag{6.23}$$

for some positive constants Γ_j, $j=1,2,\cdots,20$ and

$$r_1:=-\zeta-\frac{\zeta}{2}+\sigma+\frac{7\delta}{2}+1,\ \ r_2:=-\zeta+\sigma+\frac{5\delta}{2}+1+\nu,$$
$$r_3:=-\zeta-\frac{\zeta}{2}+\tau+\frac{7\delta}{2}+1,\ \ r_4:=-\zeta+\tau+\frac{5\delta}{2}+1+\nu,$$
$$r_5:=-2\zeta+\sigma+\frac{9\delta}{2}+1,\ \ r_6=r_7:=-\zeta-\frac{\zeta}{2}+\sigma+\frac{7\delta}{2}+1+\nu,$$
$$r_8:=-\zeta+\sigma+\frac{5\delta}{2}+1+2\nu,\ \ r_9:=2\delta-\zeta+\sigma+\hat{t}\sqrt{\eta}\delta,$$
$$r_{10}:=\delta-\zeta+\sigma+\hat{t}\sqrt{\eta}\delta,\ \ r_{11}:=2\delta-\zeta+\sigma+\hat{t}\sqrt{\eta}\delta+1,$$

$$r_{12} := 2\delta - \zeta + \tau + \hat{t}\sqrt{\eta}\delta, \;\; r_{13} := \delta - \zeta + \tau + \hat{t}\sqrt{\eta}\delta,$$

$$r_{14} := 2\delta - \zeta + \tau + \hat{t}\sqrt{\eta}\delta + 1, \;\; r_{15} := 3\delta - \zeta - \frac{\zeta}{2} + \sigma + \hat{t}\sqrt{\eta}\delta,$$

$$r_{16} := 2\delta - \zeta + \sigma - \frac{\zeta}{2} + \hat{t}\sqrt{\eta}\delta, \;\; r_{17} := 3\delta - \zeta + \sigma - \frac{\zeta}{2} + \hat{t}\sqrt{\eta}\delta + 1,$$

$$r_{18} := 2\delta - \zeta + \sigma + \nu + \hat{t}\sqrt{\eta}\delta, \;\; r_{19} := \delta - \zeta + \sigma + \nu + \hat{t}\sqrt{\eta}\delta,$$

$$r_{20} := 2\delta - \zeta + \sigma + \nu + \hat{t}\sqrt{\eta}\delta + 1.$$

Due to (6.22) all the previous constants are smaller than $-N$. Then, for the quantity $\Gamma_0 := \max_j \Gamma_j$, inequality (6.23) gives

$$|\frac{d}{dt}\mathcal{E}(t;p)| \leq \Gamma_0 q^{-N}, \;\; q \geq p_0, \tag{6.24}$$

which leads to (6.9), since the constant N is arbitrary.

The proof of the claim (6.11) follows from (6.24) in the same way as (4.22) follows from (4.38).

□

Theorem 6.3. *Consider the initial value problem* (1.5) - (1.6), *where the conditions of Theorem 6.1 and (i), (ii), (iii) of Theorem 6.2 keep in force. Assume, also, that* (4.41) *and* (6.8) *hold.*

a) If relation (6.12) *is true, then*

$$\mathcal{E}(t;p) \simeq 0, \;\; p \to +\infty, \;\; t \in Co([0, T(L_0))).$$

Moreover the growth index at infinity of the error function satisfies

$$\mathcal{G}_E(\mathcal{E}(t;\cdot)) \geq L_0, \;\; t \in Co([0, T(L_0))).$$

b) If (6.12) *keeps in force, then*

$$\frac{d}{dt}\mathcal{E}(t;p) \simeq 0, \;\; p \to +\infty, \;\; t \in Co([0, T(L_1)))$$

and

$$\mathcal{G}_E(\frac{d}{dt}\mathcal{E}(t;p)) \geq L_1, \;\; t \in Co([0, T(L_1))).$$

Proof. First of all we can see that for a fixed $\hat{t} \in (0, T(L_0))$, due to (4.41) and (4.44) we can find reals τ, σ, ν near to $-\mathcal{G}_E(\hat{x}_0)$, $-\mathcal{G}_E(x_0)$, $-\mathcal{G}_E(a(t;\cdot))$, respectively, such that

$$\exp\Big(-\frac{1}{2}\int_0^{\hat{t}} a(s;p)ds\Big) \leq p^{\frac{1}{2}\Omega(\hat{t})}.$$

Taking into account this fact and relation (6.10), we can see that

$$\max\{\tau, \sigma, \sigma+\nu\} + \frac{1}{2}\Omega(\hat{t}) < \min_{j=1}^{5} \mathcal{G}_E(\Phi_j).$$

Now, we proceed as in the proof of Theorem 6.2, where it is enough to observe that the right hand side of relation (6.20) is multiplied by the factor

$$\exp(-\frac{1}{2}\int_0^{\hat{t}} a(s;p)ds).$$

A similar procedure is followed for the proof of part (b) of the theorem. □

7. A specific case of the Initial Value Problem (1.3)-(1.4)

We shall apply the results of theorem 6.2 to a specific case of the problem (1.3) - (1.4), namely to the problem

$$x'' + 2ap^{\nu}x' - a^2p^{2\mu}x + p^m x\sin(x) = 0, \tag{7.1}$$

associated with the initial conditions

$$x(0;p) = ap^{\sigma}, \quad x'(0;p) = ap^{\tau}, \tag{7.2}$$

where, for simplicity, we have set

$$a := \frac{1}{10}, \quad \mu := 2, \quad \nu := \frac{1}{9}, \quad \tau = \sigma := \frac{1}{2}, \quad m \leq \frac{2}{9}.$$

Using these quantities we can see that all assumptions of Theorem 6.2 hold, with $E(p) = p$,

$$L_0 = \frac{19}{6}, \quad L_1 = \frac{7}{6}.$$

Then an approximate solution of the problem is given by

$$\tilde{x}t;p) := \frac{1}{10}e^{-\frac{t}{10}p^{\frac{1}{9}}}p^{\frac{1}{2}}\cosh(\frac{p^2t}{10}) + (10p^{-\frac{3}{2}} + p^{\frac{11}{18}})\sinh(\frac{p^2t}{10}), \quad t \geq 0.$$

In Figure 2 the approximate solution for the values p=1, 3.45, 5.90, 8.38, 10.80, 13.25, 15.70, 18.15 is shown.

8. Approximate solutions of the Boundary Value Problem (1.9)-(1.10)

In this section we consider Eq. (1.9) associated with the boundary conditions (1.10). Our purpose is to use the results of section 3 in order to approximate the solutions of the boundary value problem, when the parameter p approaches the critical value $+\infty$.

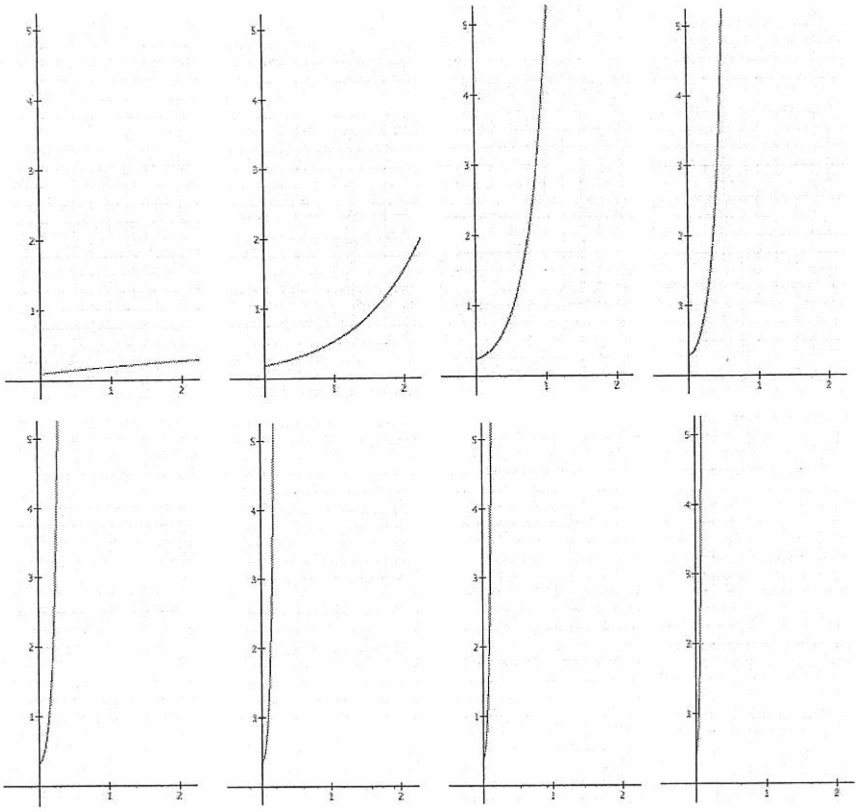

FIGURE 2. Approximate solutions of (7.1) - (7.2), when p=1, 3.45, 5.90, 8.38, 10.80, 13.25, 15.70, 18.15 respectively

To begin with define $\tau := v(1;p)$ and from now on the letter J_p will denote the interval $[0,\tau]$. Also, in order to unify our results, we make the following convention:

We shall denote by

$$S_c(v) = \begin{cases} \sin(v), & if \;\; c = +1 \\ \sinh(v), & if \;\; c = -1, \end{cases}$$

$$C_c(v) = \begin{cases} \cos(v), & if \;\; c = +1 \\ \cos(v), & if \;\; c = -1. \end{cases}$$

Our basic hypothesis which will be assumed in all the sequel without any mention is the following:

Condition 8.1. *In case $c = +1$ let*

$$\tau := v(1;p) = \int_0^1 \sqrt{b(s;p)}ds < \pi, \tag{8.1}$$

for all p large enough.

Suppose that the problem (1.9)-(1.10) admits a solution $x(t;p)$, $t \in [0,1]$. Then, Theorem 3.2 implies, and inversely, that if $y(\cdot;p)$ is a solution of equation (3.10) having boundary conditions

$$\begin{aligned} y(0;p) &= x_0(p) =: y_0(p) \\ y(\tau;p) &= y(v(1;p);p) = \frac{x(1;p)}{Y(1;p)} \\ &= x_1(p)\Big(\frac{b(1;p)}{b(0;p)}\Big)^{\frac{1}{4}} e^{\frac{1}{2}\int_0^1 a(s;p)ds} =: y_\tau(p). \end{aligned} \tag{8.2}$$

Before we seek for approximate solutions of the problem (1.9)-(1.10) we shall give conditions for the existence of solutions. To do that we need the following classical fixed point theorem:

Theorem 8.2. *(Nonlinear alternative)* [6]. *Let D be a convex subset of a Banach space X, let U be an open subset of D, and let $A : \bar{U} \to D$ be a completely continuous mapping. If $q \in U$ is a fixed element, then either A has a fixed point in $\bar{U}$, or there is a point $u \in \partial U$ and $\lambda \in (0,1)$, such that $u = \lambda Au + (1-\lambda)q$.*

To proceed we shall formulate the integral form of the problem and then we shall apply Theorem 8.2. To this end we let w be the solution of the homogeneous equation

$$w'' + cw = 0,$$

with boundary conditions $w(0;p) = y_0(p)$ and $w(\tau;p) = y_\tau(p)$. This means that w is defined as

$$w(v;p) = \frac{1}{S_c(\tau)}\big(y_0(p)(S_c(\tau - v) + y_\tau(p)S_c(v)\big). \tag{8.3}$$

(Notice that because of (8.1) in case $c = +1$ the factor $S_c(\tau)$ is positive for all τ.) Hence we see that

$$|w(v;p)| \leq q_c(|y_0| + |y_\tau|),$$

where

$$q_c := \begin{cases} \frac{1}{\min\{\sin(\sqrt{\theta}),\sin(\tau)\}}, & c = +1 \\ \frac{\sinh(\tau)}{\sinh(\sqrt{\theta})}, & c = +1. \end{cases}$$

Next we let $R(v;p)$, $v \in J$ be the solution of equation

$$R''(v;p) + cR(v;p) = H(v;p), \quad v \in J_p \tag{8.4}$$

satisfying the boundary conditions

$$R(0;p) = R(\tau;p) = 0. \tag{8.5}$$

where

$$\begin{aligned} H(v;p) :&= C(t, y(v;p);p)y(v;p) \\ &= C(t, y(v;p);p)R(v;p) + C(t, y(v;p);p)w(v;p). \end{aligned}$$

The latter, due to (3.15), implies that

$$|H(v;p)| \leq P(p)|R(v;p)| + P(p)q_c(|y_0(p)| + |y_\tau(p)|). \tag{8.6}$$

To formulate an integral form of the problem we follow an elementary method and obtain

$$R(v;p) = d_1 C_c(v) + d_2 S_c(v) + \int_0^v S_c(v-s)H(s;p)ds, \quad v \in J_p \tag{8.7}$$

for some constants d_1, d_2 to be determined from the boundary values (8.5). Thus we have

$$0 = R(0;p) = d_1$$

and

$$0 = R(\tau;p) = d_1 C_s(\tau) + d_2 S_c(\tau) + \int_0^\tau S_c(\tau - s)H(s;p)ds.$$

This implies that

$$d_2 = -\frac{1}{S_c(\tau)} \int_0^\tau S_c(\tau - s)H(s;p)ds$$

and so we have

$$R(v;p) = \int_0^\tau G(v,s;p)H(s;p)ds, \tag{8.8}$$

where the one-parameter Green's function G is defined by

$$G(v,s;p) := \frac{-S_c(v)S_c(\tau - s)}{S_c(\tau)} + S_c(v-s)\chi_{[0,v]}(s). \tag{8.9}$$

Here the symbol χ_A denotes the characteristic function of the set A. From (8.9) we can see that

$$G(v,s;p) = \begin{cases} -\frac{S_c(s)S_c(\tau - v)}{S_c(\tau)}, & 0 \leq s \leq v \leq \tau \\ -\frac{S_c(v)S_c(\tau - s)}{S_c(\tau)}, & 0 \leq v \leq s \leq \tau \end{cases}$$

From 3.1 and (8.1) it follows that for all $s, v \in [0, \tau]$ it holds

$$\max\{|G(v,s;p)|, \ |\frac{\partial}{\partial v}G(v,s;p)|\} \leq Q_c, \tag{8.10}$$

where

$$Q_c := \begin{cases} \frac{1}{\min\{\sin(\sqrt{\theta}),\sin(\tau)\}}, & c = +1 \\ \frac{(\sinh(\tau))^2}{\sinh(\sqrt{\theta})}, & c = -1. \end{cases}$$

Now we see that the operator form of the boundary value problem (3.10)-(8.2) is the following:

$$y(v;p) = w(v;p) + \int_0^\tau G(v,s;p)C(\phi(s;p),y(s;p);p)y(s;p)ds, \quad v \in J_p. \tag{8.11}$$

To show the existence of a solution of (8.11) we consider the space $C(J_p,\mathbb{R})$ of all continuous functions $y : J_p \to \mathbb{R}$ endowed with the sup-norm $\|\cdot\|$-topology. This is a Banach space. Fix a p large enough and define the operator $A : C(J_p,\mathbb{R}) \to C(J_p,\mathbb{R})$ by

$$(Az)(v) := w(v;p) + \int_0^\tau G(v,s;p)C(\phi(s;p),z(s);p)z(s)ds$$

which is completely continuous (due to Properties 3.1 and 3.3).

To proceed we assume for a moment that it holds

$$1 - P(p)\tau Q_c =: \Delta(p) = \Delta > 0, \tag{8.12}$$

where (recall that) $P(p)$ is defined in (3.15). Take any large p and let $\tau = v(1;p) =: v$. Then, clearly,

$$1 - P(p)\tau Q_c \geq \Delta > 0$$

Consider the open ball $B(0,l)$ in the space $C(J_p,\mathbb{R})$, where

$$l := \frac{\|w\|}{1 - P(p)\tau Q} + 1.$$

Here $\|w\|$ is the sup-norm of w on J_p.

Assume that the operator A does not have any fixed point in $B(0,l)$. Thus, due to Theorem 8.2 and by setting $q = 0$, there exists a point z in the boundary of $B(0,l)$ satisfying

$$z = \lambda A z,$$

for some $\lambda \in (0,1)$. This means that for each $v \in J_p$ it holds

$$|z(v)| \leq \|w\| + \int_0^\tau |G(v,s;p)||C(\phi(s;p),z(s);p)||z(s)|ds.$$

Then, from (8.10) we have

$$|z(v)| \leq \|w\| + Q_c P(p) \int_0^\tau |z(s)|ds.$$

Thus, we get

$$|z(v)| \leq \|w\| + Q_c P(p)\tau\|z\|, \tag{8.13}$$

which leads to the contradiction

$$l = \|z\| \leq \frac{\|w\|}{1 - P(p)\tau Q_c} = l - 1.$$

Taking into account the relation between the solutions of the original problem and the solution of the problem (1.9)-(1.10), as well the previous arguments, we conclude the following result:

Theorem 8.3. *If Properties 3.1, 3.3 and* (8.12) *are true, then the boundary value problem* (1.9)-(1.10) *admits at least one solution.*

Now, we give the main results of this section. First we define the function

$$\begin{aligned}
\tilde{x}(t;p) :=& \Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}} \exp\Big(-\frac{1}{2}\int_0^t a(s;p)ds\Big)\frac{1}{S_c(\int_0^1 \sqrt{b(s;p)}ds)} \\
&\times \Big\{x_0(p) S_c\Big(\int_t^1 \sqrt{b(s;p)}ds\Big) \\
&+ x_1(p)\Big(\frac{b(1;p)}{b(0;p)}\Big)^{\frac{1}{4}} e^{\frac{1}{2}\int_0^1 a(s;p)ds} S_c\Big(\int_0^t \sqrt{b(s;p)}ds\Big)\Big\} \\
=& \frac{1}{S_c(\int_0^1 \sqrt{b(s;p)}ds)}\Big\{\Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}} \\
&\times \exp\Big(-\frac{1}{2}\int_0^t a(s;p)ds\Big) S_c\Big(\int_t^1 \sqrt{b(s;p)}ds\Big)x_0(p) \\
&+ \Big(\frac{b(1;p)}{b(t;p)}\Big)^{\frac{1}{4}} e^{\frac{1}{2}\int_0^1 a(s;p)ds} S_c\Big(\int_0^t \sqrt{b(s;p)}ds\Big)\Big)x_1(p)\Big\}
\end{aligned} \tag{8.14}$$

which is going to be an approximate solution of the problem.

Theorem 8.4. *Consider the boundary value problem* (1.9) - (1.10), *where assume that Properties 3.1, 3.3, 8.1 , the conditions (i), (ii) of Theorem 4.2 and assumption* (4.41) *keep in force. Also, assume that the boundary values have a behavior like*

$$x_0, \in \mathcal{A}_E. \tag{8.15}$$

a) If the condition

$$\begin{aligned}
&\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) + \frac{3}{4}\mathcal{G}_E(b(t;\cdot)) - \Omega + \min\{\mathcal{G}_E(x_0), \mathcal{G}_E(x_1)\} \\
&\qquad =: L_0 > 0
\end{aligned} \tag{8.16}$$

is satisfied, then the existence of a solution x of the problem is guaranteed and if

$$\mathcal{E}(t;p) := x(t;p) - \tilde{x}(t;p) \tag{8.17}$$

is the error function, where $\tilde{x}$ is defined by (8.14)*, then we have*

$$\mathcal{E}(t;p) \simeq 0, \ \ p \to +\infty, \ \ t \in Co([0,1]), \tag{8.18}$$

where

$$\Omega := \frac{1}{2}\int_0^1 \omega(s)ds.$$

(Here ω is given in assumption (4.41)*.)*

Also, the growth index of the error function satisfies

$$\mathcal{G}_E(\mathcal{E}(t;\cdot)) \geq L_0, \ \ t \in Co([0,1]). \tag{8.19}$$

b) Assume that the condition

$$\begin{aligned}\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) + \frac{3}{4}\mathcal{G}_E(b(t;\cdot)) - \Omega + \min\{\mathcal{G}_E(x_0) + \frac{1}{2}\mathcal{G}_E(b(t;\cdot)),\\ \mathcal{G}_E(x_0) + \mathcal{G}_E(b(t;\cdot)), \mathcal{G}_E(x_1) + \frac{1}{2}\mathcal{G}_E(b(t;\cdot)), \mathcal{G}_E(x_0),\\ \mathcal{G}_E(x_1)\} =: L_1, \ \ t \in Co([0,1]) > 0,\end{aligned} \tag{8.20}$$

holds. Then the existence of a solution x of the problem is guaranteed and it satisfies

$$\frac{d}{dt}\mathcal{E}(t;p) \simeq 0, \ \ p \to +\infty, \ \ t \in Co([0,1]), \tag{8.21}$$

and

$$\mathcal{G}_E\Big(\frac{d}{dt}\mathcal{E}(t;\cdot)\Big) \geq L_1, \ \ t \in Co([0,1]). \tag{8.22}$$

Proof. a) Take any $N \in (0, L_0)$ and, because of (8.16), we can choose $\zeta > 0$ and real numbers μ, σ, ϱ near to $-\mathcal{G}_E(b(t;\cdot))$, $-\mathcal{G}_E(x_0)$, $-\mathcal{G}_E(x_1)$, respectively, such that

$$\min_{j=1}^{5} \lambda(\Phi_j) > \zeta \geq N + \frac{\mu}{4} + \Omega + \max\{\sigma, \varrho\}. \tag{8.23}$$

Thus, we have

$$\frac{\mu}{4} + \Omega + \max\{\sigma, \varrho\} - \zeta \leq -N \tag{8.24}$$

and, and due to Lemma 2.2,

$$P(p) \leq K(E(p))^{-\zeta}, \tag{8.25}$$

for some $K > 0$. Thus (8.12) keeps in force for p large enough. This makes Theorem 8.3 applicable and the existence of a solution is guaranteed.

Let $\mathcal{E}(t;p)$ be the error function defined in (8.17). From (8.8), (8.10) and (8.6) we have

$$|R(v;p)| \leq q_c Q_c P(p)\tau(|y_0| + |y_\tau|) + Q_c P(p) \int_0^\tau |R(s;p)|ds,$$

and therefore

$$\begin{aligned}|R(v;p)| &\leq \frac{q_c Q_c P(p)\tau(|y_0| + |y_\tau|)}{1 - Q_c P(p)\tau} \\ &\leq \frac{1}{\Delta} q_c Q_c P(p)\tau(|y_0| + |y_\tau|), \quad v \in J_p.\end{aligned} \tag{8.26}$$

Then observe that

$$\begin{aligned}|\mathcal{E}(t;p)| &= |x(t;p) - Y(t;p)w(v(t;p);p)| \\ &= |Y(t;p)||y(v(t;p);p) - w(v(t;p);p)| = |Y(t;p)||R(v(t;p);p)|,\end{aligned}$$

because of (4.4). Thus, from (8.26) it follows that for all $t \in [0,1]$ it holds

$$\begin{aligned}|\mathcal{E}(t;p)| &\leq \Delta^{-1}|Y(t;p)|q_c Q_c P(p)\tau(|y_0| + |y_\tau|) \\ &= \Delta^{-1}\Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}} e^{-\frac{1}{2}\int_0^t a(s;p)ds} q_c Q_c P(p)\tau\big(|y_0| + |y_\tau|\big) \\ &= \Delta^{-1} q_c Q_c \sqrt{\|b(\cdot;p)\|}P(p) \\ &\times \Big[\Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}} e^{-\frac{1}{2}\int_0^t a(s;p)ds}|x_0(p)| \\ &+ \Big(\frac{b(1;p)}{b(t;p)}\Big)^{\frac{1}{4}} e^{\frac{1}{2}\int_t^1 a(s;p)ds}|x_1(p)|\Big].\end{aligned} \tag{8.27}$$

From (8.25) and (8.27) for all large p (especially for all p with $q := E(p) > 1$) it follows that

$$\begin{aligned}|\mathcal{E}(t;p)| &\leq \Delta^{-1} q_c Q_c \tau K q^{-\zeta} \\ &\times \frac{K_1^{\frac{3}{4}} q^{\frac{3\mu}{4}}}{\theta^{\frac{1}{4}}} \exp\Big(\log(q)\frac{1}{2}\int_0^1 \omega(s)ds\Big)\Big(K_2 q^\sigma + K_3 q^\varrho\Big) \\ &\leq K_4 q^{-\zeta + \frac{3\mu}{4} + \Omega}(K_2 q^\sigma + K_3 q^\varrho).\end{aligned}$$

Finally, from (8.24) we get

$$|\mathcal{E}(t;p)| \leq \hat{K} q^{-N}, \tag{8.28}$$

for some $\hat{K} > 0$, which, obviously, leads to (8.18). Relation (8.19) follows from (8.28) as exactly relation (4.22) follows from (4.38).

b) Next consider the first order derivative of the error function $\mathcal{E}(t;p)$. Due to (8.20), given any small ε and $N \in (0, L_1 - \varepsilon)$, we get reals $\zeta >$

0 and real $\mu, \nu, \sigma, \varrho > 0$, near to $-\mathcal{G}_E(b(t;\cdot))$, $-\mathcal{G}_E(a(t;\cdot))$, $-\mathcal{G}_E(x_0)$, $\mathcal{G}_E(x_1)$, respectively, such that

$$\begin{aligned}\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) > \zeta > N_1 + \frac{3\mu}{4} + \Omega + \max\{\sigma + \frac{\mu}{2}, \sigma + \nu, \\ \varrho + \frac{\mu}{2}, \varrho + \nu, \mu + \varrho, \mu + \sigma\}.\end{aligned} \tag{8.29}$$

From (8.9) and (8.10) we observe that it holds

$$\begin{aligned}|\frac{d}{dv}R(v;p)| &= |\frac{d}{dv}\int_0^\tau G(v,s;p)H(s;p)ds| \\ &\leq Q_c\tau(P(p)|R(v;p)| + P(p)q_c(|y_0| + |y_\tau|) \\ &\leq q_cQ_c\tau P(p)[\Delta^{-1}Q_c\tau P(p) + 1](|y_0| + |y_\tau|).\end{aligned}$$

From this relation it follows that

$$\begin{aligned}|\frac{d}{dt}\mathcal{E}(t;p)| &= |\frac{d}{dt}Y(t;p)R(v(t;p);p) + Y(t;p)\frac{d}{dv}R(v(t;p);p)\frac{d}{dt}v(t;p)| \\ &\leq |Y(t;p)|\Big\{\Big(\frac{\sqrt{\Phi_1(p)b(t;p)}}{4} + \frac{|a(t;p)|}{2}\Big)|R(v(t;p);p)| \\ &+ |\frac{d}{dv}R(v(t;p);p)|\sqrt{b(t;p)}\Big\} \\ &\leq |Y(t;p)|\Big\{\Big(\frac{\sqrt{\Phi_1(p)b(t;p)}}{4} + \frac{|a(t;p)|}{2}\Big) \\ &\times \Delta^{-1}q_cQ_cP(p)\tau(|y_0| + |y_\tau|) \\ &+ \sqrt{b(t;p)}q_cQ_c\tau P(p)[\Delta^{-1}Q_c\tau P(p) + 1](|y_0| + |y_\tau|)\Big\}.\end{aligned}$$

Therefore, for all large p (especially for p with $q := E(p) > 1$) we obtain

$$\begin{aligned}|\frac{d}{dt}\mathcal{E}(t;p)| &\leq q_cQ_c\hat{\tau}P(p)\Big[|x_0(p)|\Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}}e^{-\int_0^t a(s;p)ds} \\ &+ |x_1(p)|\Big(\frac{b(1;p)}{b(t;p)}\Big)^{\frac{1}{4}}e^{\int_t^1 a(s;p)ds}\Big] \\ &\times \Big\{\Big(\frac{\sqrt{\Phi_1(p)b(t;p)}}{4} + \frac{|a(t;p)|}{2}\Big)\Delta^{-1} \\ &+ \sqrt{b(t;p)}[\Delta^{-1}Q_c\tau P(p) + 1]\Big\} \\ &\leq q^{-\zeta+\Omega+\frac{3\mu}{4}}(M_1q^{\sigma+\frac{\mu}{2}} + M_2q^{\sigma+\nu} \\ &+ M_3q^{\rho+\frac{\mu}{2}} + M_4q^{\rho+\nu} + M_5q^{\varrho+\mu} + M_6q^{\varrho+\sigma}),\end{aligned} \tag{8.30}$$

for some positive constants $M_1, M_2, M_3, M_4, M_5, M_6$ not depending on the parameter p. Taking into account the condition (8.29) we conclude that

$$|\frac{d}{dt}\mathcal{E}(t;p)| \leq Mq^{-N_1},$$

for all large p. Now, the rest of the proof follows as previously. □

From inequalities (8.27) and (8.30) we can easily see that if the function $a(\cdot;p)$ is non-negative uniformly for all p and $x_1(p) = 0$, or $a(\cdot;p)$ is non-positive uniformly for all p and $x_0(p) = 0$, then the conditions of Theorem 8.4 can be weaken. Indeed, we have the following results, whose the proofs follow the same lines as in Theorem 8.4:

Theorem 8.5. *Consider the boundary value problem* (1.9) - (1.10), *where assume that Properties 3.1, 3.3, 8.1 and the conditions (i), (ii) of Theorem 4.2 hold.*

Also, assume that $a(t;p) \geq 0$ [*respectively* $a(t;p) \leq 0$,] *for all* $t \in [0,1]$ *and* p *large, as well as*

$$x_0 \in \mathcal{A}_E \text{ and } x_1(p) = 0, \text{ for all large } p$$

[*resp.*

$$x_0(p) = 0, \text{ for all large } p \text{ and } x_1(p) \in \mathcal{A}_E].$$

a) If the condition

$$\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) + \frac{1}{4}\mathcal{G}_E(b(t;\cdot)) + \mathcal{G}_E(x_0) =: L_0 > 0$$

[*resp.*

$$\min_{j=1}^{5} \mathcal{G}_E(\Phi_j)\frac{1}{4}\mathcal{G}_E(b(t;\cdot)) + \mathcal{G}_E(x_1) = L_0 > 0]$$

is satisfied, then the existence of a solution x of the problem is guaranteed and if

$$\mathcal{E}(t;p) = x(t;p) - \tilde{x}(t;p)$$

is the error function, where $\tilde{x}$ is defined by (8.14), *then* (8.18) *holds.*

Also, the growth index at infinity of the error function satisfies (8.19).

b) If the condition

$$\begin{aligned}\min_{j=1}^{5} \lambda(\Phi_j) &+ \frac{1}{4}\mathcal{G}_E(b(t;\cdot)) + \mathcal{G}_E(x_0) \\ &+ \min\{\frac{1}{2}\mathcal{G}_E(b(t;\cdot)), \mathcal{G}_E(a(t;\cdot))\} =: L_1 > 0\end{aligned} \tag{8.31}$$

[*resp.*

$$\begin{aligned}\min_{j=1}^{5} \lambda(\Phi_j) &+ \frac{1}{4}\mathcal{G}_E(b(t;\cdot)) + \mathcal{G}_E(x_1) \\ &+ \min\{\frac{1}{2}\mathcal{G}_E(b(t;\cdot), \mathcal{G}_E(a(t;\cdot))\} =: L_1 > 0]\end{aligned}$$

holds, then the existence of a solution x of the problem is guaranteed and it satisfies (8.21) *and* (8.22).

9. Applications

1. Consider the equation

$$x'' + \frac{2}{\sin(1)}\cos(t)\log(p)x' - [1+p^{10}]x + p^{-1}x\sin(x) = 0, \tag{9.1}$$

associated with boundary values

$$x_0(p) = \frac{1}{5}p, \quad x_1(p) = \frac{1}{10}(1+\frac{1}{p}). \tag{9.2}$$

Conditions (3.11), (3.12) and (3.13) are satisfied, if we get the functions

$$\Phi_1(p) = \Phi_2(p) = \Phi_3(p) = \Phi_4(p) = k_1 p^{-\frac{39}{4}}$$

and

$$\Phi_5(p) := k_2 p^{-10},$$

for some $k_1, k_2 > 0$. So case (a) of Theorem 8.4 is applicable with $E(p) := p$. It is not hard to see that an approximate solution of the problem is the function

$$\tilde{x}(t;p) := e^{-\frac{\sin(t)}{\sin(1)}}\Big[p\frac{\sinh\left((1-t)\sqrt{1+p^{10}}\right)}{\sinh\left(\sqrt{1+p^{10}}\right)} + e(p+\frac{1}{p})\frac{\sinh\left(t\sqrt{1+p^{10}}\right)}{\sinh\left(\sqrt{1+p^{10}}\right)}\Big],$$

satisfying

$$\mathcal{G}_E\big(x(t;\cdot) - \tilde{x}(t;\cdot)\big) \geq \frac{1}{4}.$$

The function for the values of $p = 1, 1.5, 2, 2.5$ has a graph shown in Figure 3.

2. Consider the equation

$$x'' + \frac{2}{\sqrt{p}}x' + [\frac{\pi}{4} + p^{-0.1}]x + \frac{x\sin(x)}{p} = 0, \tag{9.3}$$

associated with boundary values

$$x_0(p) = 0.2\sqrt{p}, \quad x_1(p) = 0. \tag{9.4}$$

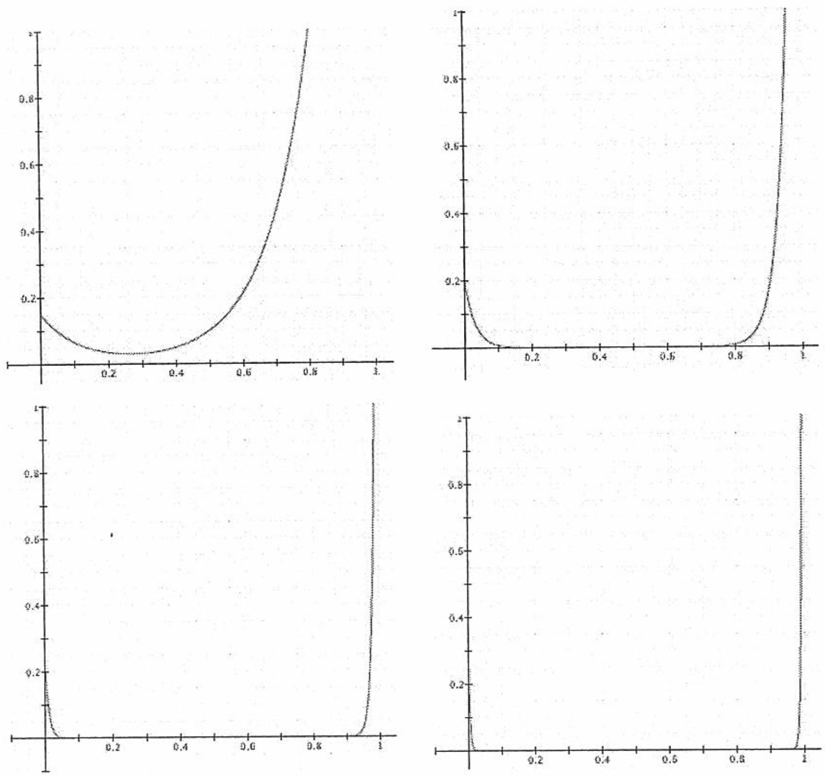

FIGURE 3. Approximate solutions of (9.1) - (9.2), when p=1, 1.5, 2, 2.5, respectively

We can take $E(p) := p$ and

$$\Phi_1(p) = \Phi_2(p) = \Phi_3(p) = \Phi_4(p) = \Phi_5(p) := k_1 p^{-0.9}.$$

Then conditions (3.11), (3.12) and (3.13) are satisfied and so Theorem 8.4 is applicable with $L_0 = \frac{3}{8}$ and $L_1 = \frac{23}{40}$.. In this case it is not hard to see that an approximate solution of the problem is the function defined on the interval $[0, 1]$ by the type

$$\tilde{x}(t;p) := 0.1\sqrt{p}(1 + \cos(15\sqrt{t}) \exp\left(\frac{-t}{\sqrt{p}}\right) \frac{\sin\left((1-t)\sqrt{\frac{\pi}{4} + p^{-0.1}}\right)}{\sin\left(\sqrt{\frac{\pi}{4} + p^{-0.1}}\right)}.$$

The graph of this function for the values of $p = 4, 10, 20, 30$ is shown in Figure 4

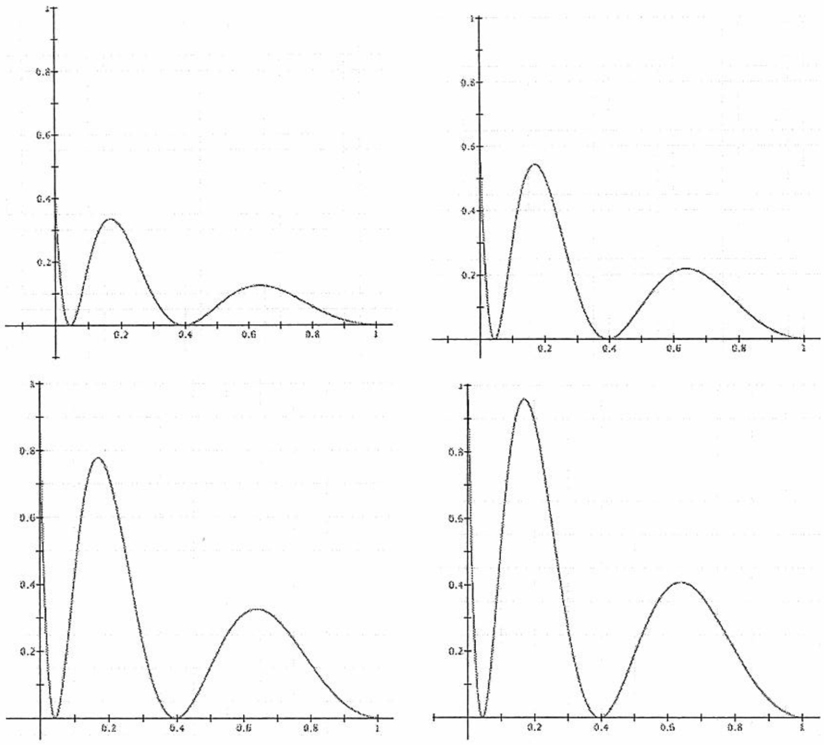

FIGURE 4. Approximate solutions of (9.3) - (9.4), when p=4, 10, 20, 30 respectively

10. APPROXIMATE SOLUTIONS OF THE BOUNDARY VALUE PROBLEM (1.9)-(1.8)

In this section we shall discuss the approximate solutions of the problem (1.9) - (1.8). We shall use the results of section 3 to obtain approximate solutions when the parameter p tends to $+\infty$. Again, as in section 8 we define $\tau := v(1;p)$, $J_P := [0,\tau]$ and use the symbols S_c and C_c.

Our basic hypothesis which will be assumed in all the sequel without any mention is that Properties 3.1 and 3.3 will keep in force for all $t \in [0,1]$.

Assume that equation (1.9) admits a solution satisfying the conditions

$$x(0;p) = x_0 \text{ and } x(1;p) = m(p)x(\xi;p),$$

for a certain point $\xi \in [0,1)$ and a real number $m(p)$. Then Theorem 3.2 implies that a function $x(\cdot;p)$ is a solution of the problem, if and only if $y(\cdot;p)$ is a solution of equation (3.10) and boundary conditions

$$\begin{aligned} y(0;p) &= x_0(p) =: y_0(p) \\ y(\tau;p) &= y(v(1);p) = \frac{x(1;p)}{Y(1;p)} = m(p)\frac{x(\xi;p)}{Y(1;p)} \\ &= m(p)\frac{Y(\xi;p)}{Y(1;p)}y(v(\xi;p);p) =: m^*(p)y(v(\xi;p);p). \end{aligned} \tag{10.1}$$

Before we seek for approximate solutions of the problem (1.9)-(1.8) we shall impose conditions for the existence of solutions. To do that we shall use, again, the Fixed Point Theorem 8.2. To proceed we assume the following:

Condition 10.1. *i) There is some $\rho > 0$ such that*

$$\frac{S_c(\int_0^\xi \sqrt{b(s;p)}ds}{S_c(\int_0^1 \sqrt{b(s;p)}ds} \geq \rho,$$

for all p large enough.

ii) It happens

$$\lim_{p\to+\infty} m(p) = +\infty.$$

iv) There is some $\bar{a} > 0$ such that

$$0 \leq a(t;p) \leq 2\bar{a},$$

for all $t \in [0,1]$ and p large enough.

iii) There are $\theta, b_0 > 0$ such that

$$\theta \leq b(t;p) \leq b_0$$

for all $t \in (0,1)$ and p large enough.

Before we seek for approximate solutions of the problem (3.10)-(10.1), we shall investigate the existence of solutions.

Let w solve the equation $w'' + cw = 0$ and satisfies the conditions

$$w(0;p) = y_0(p)$$

and

$$w(\tau;p) = m^*(p)w(v(\xi;p);p).$$

Solving this problem we obtain

$$w(v;p) = \frac{S_c(\tau - v) - m^* S_c(v(\xi;p) - v)}{S_c(\tau) - m^* S_c(v(\xi;p))} y_0(p). \tag{10.2}$$

We shall show that the solution w is bounded. Indeed, from (10.2) we observe that

$$|w(v;p)| \leq \frac{S_c(\tau) + m^* S_c(\tau)}{m^* S_c(v(\xi;p)) - S_c(\tau)} |y_0(p)|$$

and by using the bounds of all arguments involved we obtain

$$|w(v;p)| \leq \frac{m(p)\left(\frac{b(1;p)}{b(\xi;p)}\right)^{\frac{1}{4}} e^{\frac{1}{2}\int_0^1 a(s;p)ds} + 1}{m(p)\left(\frac{b(1;p)}{b(\xi;p)}\right)^{\frac{1}{4}} \frac{S_c(\int_0^\xi \sqrt{b(s;p)}ds)}{S_c(\int_0^1 \sqrt{b(s;p)}ds)} - 1} |y_0(p)|.$$

Hence, because of Condition 10.1, we obtain

$$|w(v;p)| \leq \frac{m(p)\sqrt{b_0}e^{\bar{a}} + (b_0\theta)^{\frac{1}{4}}}{m(p)\sqrt{\theta}\rho - (b_0\theta)^{\frac{1}{4}}} |y_0(p)| \leq \rho_0 |y_0(p)|, \tag{10.3}$$

for all large p, where

$$\rho_0 := \Big(\frac{\sqrt{b_0}e^{\bar{a}}}{\sqrt{\theta}\rho} + 1\Big).$$

As in previous sections, we set $R := y - w$. We shall search for constants d_1 and d_2 such that the function

$$R(v;p) := d_1 C_c(v) + d_2 S_c(v) + \int_0^v S_c(v-s)H(s;p)ds$$

be a solution of the nonhomogeneous equation

$$R'' + cR = H$$

satisfying the conditions

$$R(0;p) = 0 \text{ and } R(\tau;p) = y(\tau;p) - w(\tau;p) = m^* R(v(\xi;p)). \tag{10.4}$$

Here H is the function defined by

$$H(t;p) := C(t, y(v;p);p)R(v;p) + C(t, y(v;p);p)w(v;p),$$

which, due to (10.3), satisfies the inequality

$$|H(v;p)| \leq P(p)|R(v;p)| + P(p)\rho_0|y_0(p)|. \tag{10.5}$$

Then we obtain that

$$d_1 = 0$$

and

$$d_2 = \frac{1}{S_c(\tau) - m^*(p)S_c(v(\xi;p))} \Big[\int_0^{v(\xi;p)} S_c(v(\xi;p)) - s)H(s;p)ds - \int_0^\tau S_c(\tau - s)H(s;p)ds\Big].$$

Therefore the solution $R(v;p)$ takes the form

$$R(v;p) = \frac{S_c(v)}{S_c(\tau) - m^*(p)S_c(v(\xi;p))}\Big[\int_0^{v(\xi;p)} S_c(v(\xi;p)) - s)H(s;p)ds$$
$$- \int_0^{\tau} S_c(\tau - s)H(s;p)ds\Big] + \int_0^{v} S_c(v-s)H(s;p)ds,$$

namely

$$R(v;p) = \int_0^{\tau} G(v,s;p)H(s;p)ds,$$

where the Green's function G is defined by

$$G(v,s;p) := \begin{cases} \frac{S_c(v)\left[S_c(v_\xi - s) - S_c(\tau - s)\right]}{S_c(\tau) - m^*(p)S_c(v(\xi;p))} + S_c(v-s), \\ \qquad\qquad 0 \leq s \leq v_\xi < v \\ -\frac{S_c(v)S_c(\tau - s)}{S_c(\tau) - m^*(p)S_c(v(\xi;p))} + S_c(v-s), \\ \qquad\qquad 0 \leq v_\xi < s < v \\ -\frac{S_c(v)S_c(\tau - s)}{S_c(\tau) - m^*(p)S_c(v(\xi;p))}, \qquad 0 \leq v_\xi < v < s. \end{cases}$$

To obtain upper C^1 bounds of the kernel G we distinguish the following cases:

$0 \leq s \leq v_\xi \leq v$.
In this case for p large enough it holds

$$|G(v,s;p)| \leq \frac{2(S_c(\tau))^2}{m^*(p)S_c(v(\xi;p)) - S_c(\tau)} + S_c(\tau)$$
$$\leq \frac{2S_c\big(\int_0^1 \sqrt{b(s;p)}ds\big)}{m(p)\Big(\frac{b(1;p)}{b(\xi;p)}\Big)^{\frac{1}{4}} e^{\frac{1}{2}\int_0^1 a(s)ds} \frac{S_c\big(\int_0^\xi \sqrt{b(s;p)}ds\big)}{S_c\big(\int_0^1 \sqrt{b(s;p)}ds\big)} - 1}$$
$$+ S_c\Big(\int_0^1 \sqrt{b(s;p)}ds\Big).$$

Thus due to Condition 10.1 there exists some $\hat{p}$ such that for all $p \geq \hat{p}$ it holds

$$|G(v,s;p)| \leq \Big[\frac{2}{m(p)(\frac{\theta}{b_0})^{\frac{1}{4}}\rho - 1} + 1\Big]k_1 \leq 2k_1,$$

where

$$k_1 := \begin{cases} e^{\sqrt{b_0}}, & c = -1 \\ 1, & c = 1. \end{cases}$$

Also, we can easily see that, for large enough p the first partial derivative of G (with respect to v) satisfies

$$\begin{aligned}\Big|\frac{\partial}{\partial v}G(v,s;p)\Big| &\leq \frac{S_c(\tau)C_c(\tau)}{m^*(p)S_c(v(\xi;p))-S_c(\tau)}+C_c(\tau)\\ &\leq \frac{C_c\big(\int_0^1\sqrt{b(s;p)}ds\big)}{m(p)\Big(\frac{b(1;p)}{b(\xi;p)}\Big)^{\frac{1}{4}}e^{\frac{1}{2}\int_0^1 a(s)ds}\frac{S_c\big(\int_0^\xi\sqrt{b(s;p)}ds\big)}{S_c\big(\int_0^1\sqrt{b(s;p)}ds\big)}-1}\\ &+C_c\Big(\int_0^1\sqrt{b(s;p)}ds\Big)\leq\frac{2k_1}{m(p)(\frac{\theta}{b_0})^{\frac{1}{4}}\rho-1}+2k_1\\ &=2k_1\Big[\frac{1}{m(p)(\frac{\theta}{b_0})^{\frac{1}{4}}\rho-1}+1\Big]\leq 4k_1.\end{aligned}$$

$0\leq v_\xi\leq s\leq v$.

In this case for p large enough it holds

$$\begin{aligned}|G(v,s;p)| &\leq \frac{(S_c(\tau))^2}{m^*(p)S_c(v(\xi;p))-S_c(\tau)}+S_c(\tau)\\ &\leq \frac{S_c\big(\int_0^1\sqrt{b(s;p)}ds\big)}{m(p)\Big(\frac{b(1;p)}{b(\xi;p)}\Big)^{\frac{1}{4}}e^{\frac{1}{2}\int_0^1 a(s)ds}\frac{S_c\big(\int_0^\xi\sqrt{b(s;p)}ds\big)}{S_c\big(\int_0^1\sqrt{b(s;p)}ds\big)}-1}\\ &+S_c\Big(\int_0^1\sqrt{b(s;p)}ds\Big)\leq\cdots\leq 2k_1.\end{aligned}$$

Similarly, we can obtain that for $0\leq v_\xi\leq s\leq v$ and p large enough, it holds

$$|G(v;s;p)|\leq 2k_1 \text{ and } \Big|\frac{\partial}{\partial v}G(v,s;p)\Big|\leq 4k_1,$$

while, for $0\leq v_\xi\leq v\leq s$, it holds

$$|G(v;s;p)|\leq k_1 \text{ and } \Big|\frac{\partial}{\partial v}G(v,s;p)\Big|\leq 2k_1.$$

Therefore for all s,v we have

$$\max\{|G(v,s;p)|,\ \Big|\frac{\partial}{\partial v}G(v,s;p)\Big|\}\leq 4k_1. \tag{10.6}$$

Applying the previous arguments we obtain that

$$|R(v;p)|\leq\frac{4k_1\rho_0 b_0^{\frac{1}{2}}}{\Delta_1}P(p)|x_0(p)|. \tag{10.7}$$

Here Δ is defined as

$$\Delta:=1-4k_1P(p)b_0^{\frac{1}{2}}=:\Delta_1(p)>0, \tag{10.8}$$

where $P(p)$ is defined in (3.15).

Hence the operator form of the boundary value problem (3.10)-(10.1) is the following:

$$\begin{aligned} y(v;p) &= w(v;p) \\ &+ \int_0^\tau G(v,s;p)C(\phi(s;p),y(s;p);p)y(s;p)ds, \quad v \in J_p. \end{aligned} \tag{10.9}$$

To show the existence of a solution of (10.9), as in Section 8, we consider the Banach space $C(J_p,\mathbb{R})$ of all continuous functions $y : J_p \to \mathbb{R}$ endowed with the sup-norm $\|\cdot\|$-topology. Fix a p large enough and define the operator $A : C(J_p,\mathbb{R}) \to C(J_p,\mathbb{R})$ by

$$(Az)(v) := w(v;p) + \int_0^\tau G(v,s;p)C(\phi(s;p),z(s);p)z(s)ds$$

which is completely continuous (due to Properties 3.1 and 3.3).

To proceed we assume for a moment that it holds

Take a large enough p and set $\tau = v(1;p) =: v$. Then we have $v \le b_0^{\frac{1}{2}}$ and so it holds

$$1 - 4k_1P(p)\tau \ge \Delta_1 > 0.$$

Consider the open ball $B(0,l_1)$ in the space $C(J,\mathbb{R})$, where

$$l_1 := \frac{\|w\|}{1 - 4k_1P(p)\tau} + 1.$$

As in Section 8, assume that the operator A does not have any fixed point in $B(0,l_1)$. Thus, due to Theorem 8.2 and by setting $q = 0$, there exists a point z in the boundary of $B(0,l_1)$ satisfying

$$z = \lambda Az,$$

for some $\lambda \in (0,1)$. This means that for each $v \in J_p$ it holds

$$|z(v)| \le \|w\| + \int_0^\tau |G(v,s;p)||C(\phi(s;p),z(s);p)||z(s)|ds.$$

Then we have

$$|z(v)| \le \|w\| + 4k_1P(p)\int_0^\tau |z(s)|ds.$$

and therefore

$$|z(v)| \le \|w\| + 4k_1P(p)\tau\|z\|,$$

which leads to the contradiction

$$l_1 = \|z\| \le \frac{\|w\|}{1 - 4k_1P(p)\tau} = l_1 - 1.$$

Taking into account the relation between the solutions of the original problem and the solution of the problem (1.9)-(1.8), as well the previous arguments, we conclude the following result:

Theorem 10.2. *If Properties 3.1, 3.3 and* (10.8) *are true, then the boundary value problem* (1.9)-(1.8) *admits at least one solution.*

Now, we give the main results of this section. If w is the function defined in (10.2) we define the function

$$\begin{aligned}\tilde{x}(t;p) :&= Y(t;p)w(v(t;p);p)\\ &= Y(t;p)\frac{S_c(\tau - v) - m^* S_c(v(\xi;p) - v)}{S_c(\tau) - m^* S_c(v(\xi;p))}y_0(p)\\ &= \Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}} \exp\Big(-\frac{1}{2}\int_0^t a(s;p)ds\Big)\frac{X(t;p)}{X(0;p)}x_0(p),\end{aligned} \tag{10.10}$$

where

$$\begin{aligned}X(t;p) :&= S_c\Big(\int_t^1 \sqrt{b(s;p)}ds\Big)\\ &\quad - m(p)\Big(\frac{b(1;p)}{b(\xi;p)}\Big)^{\frac{1}{4}} e^{\frac{1}{2}\int_\xi^1 a(s)ds} S_c\Big(\int_t^\xi \sqrt{b(s;p)}ds\Big),\end{aligned}$$

which, as we shall show, it is an approximate solution of the problem under discussion.

Theorem 10.3. *Consider the boundary value problem* (1.9)-(1.8), *where assume that Properties 3.1, 3.3, 8.1 , the conditions* (10.8) *and (i), (ii) of Theorem 4.2 keep in force. Also, assume that* $x_0 \in \mathcal{A}_E$.

a) If the condition

$$\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) + \mathcal{G}_E(x_0) =: L > 0 \tag{10.11}$$

is satisfied, then the existence of a solution x of the problem is guaranteed and if

$$\mathcal{E}(t;p) := x(t;p) - \tilde{x}(t;p)$$

is the error function, where $\tilde{x}$ is defined by (8.14), *then we have*

$$\mathcal{E}(t;p) \simeq 0, \ \ p \to +\infty, \ \ t \in Co([0,1]). \tag{10.12}$$

Also, the growth index at infinity of the error function satisfies

$$\mathcal{G}_E(\mathcal{E}(t;\cdot)) \geq L, \ \ t \in Co([0,1]). \tag{10.13}$$

b) Moreover we have

$$\frac{d}{dt}\mathcal{E}(t;p) \simeq 0, \ \ p \to +\infty, \ \ t \in Co([0,1]), \tag{10.14}$$

and

$$\mathcal{G}_E\Big(\frac{d}{dt}\mathcal{E}(t;\cdot)\Big) \geq L, \ \ t \in Co([0,1]). \tag{10.15}$$

Proof. a) Take a $N \in (0, L)$ and choose $\zeta > 0$ as well as $-\sigma < \mathcal{G}_E(x_0)$, (thus we have

$$|x_0(p)| \leq K_3(E(p))^{\sigma},$$

for some $K_3 > 0$) such that

$$\min_{j=1}^{5} \lambda(\Phi_j) > \zeta \geq N + \sigma. \tag{10.16}$$

Therefore it follows that

$$\sigma - \zeta \leq -N \tag{10.17}$$

and

$$P(p) \leq K(E(p))^{-\zeta}, \tag{10.18}$$

for some $K > 0$. Thus (10.8) keeps in force for p large enough. This makes Theorem 10.2 applicable and the existence of a solution is guaranteed.

Let $\mathcal{E}(t;p)$ be the error function defined in (8.17). From (10.7) it is easy to obtain that

$$|\mathcal{E}(t;p)| \leq \Lambda_1(E(p))^{\sigma-\zeta}.$$

for all large p, for some $\Lambda_1 > 0$. Obviously, this relation implies (10.12) as well as (10.13).

b) Next consider the first order derivative of the error function $\mathcal{E}(t;p)$. Again, as above, we obtain

$$\begin{aligned}|\frac{d}{dt}R(v(t;p);p)| &= |\frac{d}{dv}\int_0^{\mathcal{T}} G(v,s;p)H(s;p)ds\frac{d}{t}v(t;p)| \\ &\leq Y(t;p)\Big[\frac{1}{4}\sqrt{\Phi_1(p)b(0;p)} + \frac{a(t;p)}{2} \\ &+ \int_0^{\mathcal{T}} \Big(G(v,s;p)| + |\frac{d}{dt}v(t;p)||\frac{\partial}{\partial v}G(v,s;p)|\Big)|H(s;p)|ds\Big].\end{aligned}$$

Now, we use (10.16), (10.18), (10.17), (10.6), (10.5) and (10.7) to conclude that for some positive constants k_3, k_4 it holds

$$|\frac{d}{dt}\mathcal{E}(t;p)| \leq k_3P(p)|x_0(p)| \leq k_4(E(p))^{\sigma-\zeta} < k_4(E(p))^{-N},$$

from which the result follows. □

11. An application

Consider the equation

$$x'' + x' + x + \frac{x\sin(x)}{p} = 0, \quad t \in [0,1] \tag{11.1}$$

associated with the following boundary value conditions:

$$x(0;p) = p^{-1}, \quad x(1;p) = e^p x(\frac{1}{2};p). \tag{11.2}$$

We can easily see that with respect to the unbounded function $E(p) :=$ p we have

$$\mathcal{G}_E(\Phi_j) = 1, \quad j = 1,2,3,4,5 \text{ and } \mathcal{G}_E(x_0) = 2.$$

Therefore $L = 2$ and, so, Theorem 10.3 applies. This means that there is a solution of the problem (11.1)-(11.2) and an approximate solution of it is the following (according to (10.10)):

$$\tilde{x}(t;p) := \frac{\sin(1-t) - e^p e^{\frac{1}{4}} \sin(\frac{1}{2} - t)}{\sin(1) - e^p e^{\frac{1}{4}} \sin(\frac{1}{2})} e^{-\frac{t}{2}} p^{-2}, \quad t \in [0,1].$$

The graph of this function for the values of $p = 3.83, 6.33, 8.83, 15.50$ is shown in Figure 5

12. Discussion

We have presented a method of computing the approximate solutions of two initial value problems and two boundary value problems concerning the second order ordinary differential equation (1.5). First of all in section 2 we have given the meaning of measuring the approximation, by introducing the growth index of a function. It is proved that this meaning helps a lot to get information on how close to the actual solution is the approximate solution as the parameter p tends to $+\infty$. Section 3 of the work provided the first step of the method, since therein we have shown the way of transforming by (3.1) the original equation to an auxiliary and easy to elaborate differential equation (3.10).

The sign of the response coefficient $b(t;p)$ plays an essential role. If it is positive, we have an wave featured solution, while in case it is negative we have exponential picture. This is the reason for discussing the two cases separately especially in the initial value problems. The first case is exhibited in Section 4, where in Theorem 4.1 we show first the existence of a solution of the initial value problem and prepare the ground for the existence of C^1-approximate solutions provided in Theorems 4.2 and Theorem 4.3. The two theorems give, mainly, similar

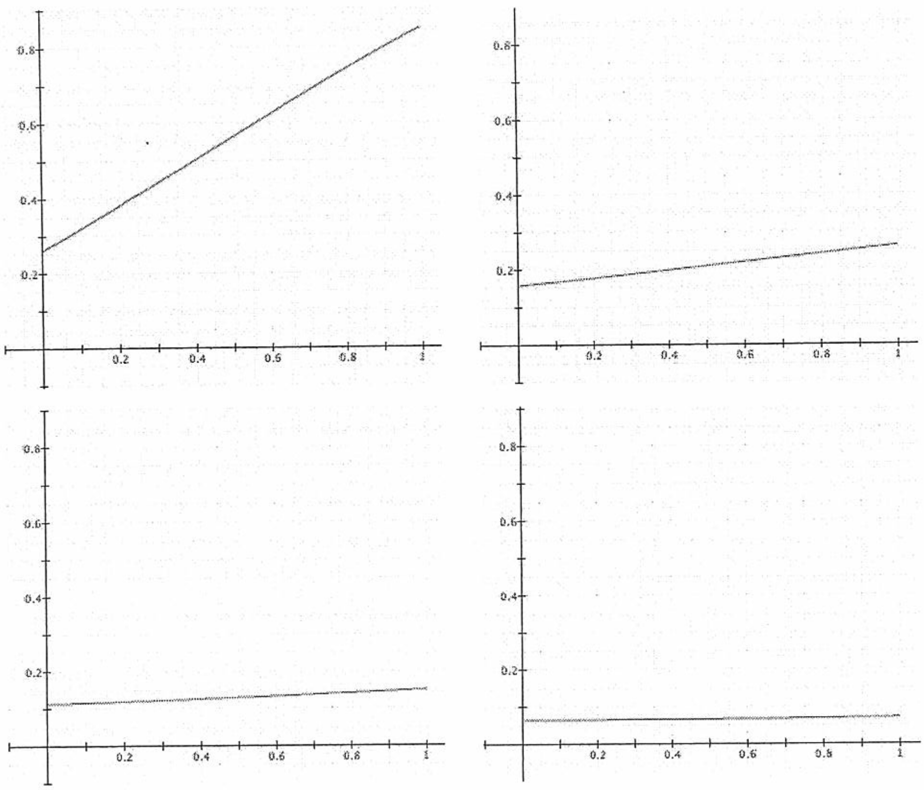

FIGURE 5. Approximate solutions of (11.1) - (11.2), when p=3.83, 6.33, 8.83, 15.50, respectively.

results, but in the first theorem we assumed that the coefficient $a(t;p)$ is positive and in the second it is assumed that it may take negative values as well.

An application of the results in an example where the two coefficients $a(t;p)$ and $b(t;p)$ are positive, is given in Section 5, where the C^1-approximate solution is computed. The case of negative $b(t;p)$ is discussed in section 6 and the approximate results are applied to a initial value problem in Section 7.

The boundary value problem (1.9)-(1.10) is discussed in Section 8. First by the help of the (Fixed Point Theorem of) Nonlinear Alternative we have guaranteed in Theorem 8.3 the existence of solutions of the problem. Then, in Theorem 8.4 we gave estimates of the error function $\mathcal{E}(t;p) := x(t;p) - \tilde{x}(t;p)$, where $\tilde{x}(t;p)$ is the C^1-approximate solution. Here we are able to give simultaneously our results in the cases of positive and negative $b(t;p)$. A specific case when $a(t;p)$ is nonnegative and the solution vanishes in an edge of the existence interval is discussed separately in Theorem 8.5, while two applications of the results where given in Section 9.

In Section 10 we investigated the boundary value problem (1.9)-(1.8). Again, first in Theorem 10.2 we solved the existence problem by using the Nonlinear Alternative and then we proceeded to the proof of the existence of C^1-approximate solutions in Theorem 10.3. An application to specific equation is given in the last section 11.

Notice that all examples which we have presented are associated with some pictures[2], which show the change of the approximate solutions, as the parameter p takes large values and tends to $+\infty$.

As we have seen, in order to apply the method to a problem we have to do two things: First to transform the original equation to a new one and then to transform the initial values or the boundary values to the new ones. Both of them are important in the process of the method.

And as the transformation of the original equation was already given in (3.10), what one has to do is to proceed to the transformation of the boundary values. For instance, in case the boundary values of the original problem are of the form

$$x(0;p) = x'(0;p), \quad x(1;p) = x'(1;p),$$

then, it is not hard to show that, under the transformation S_p the new function $y(\cdot;p)$ is required to satisfy the boundary values

$$y'(0;p) = \frac{1}{\sqrt{b(0;p)}}\Big[1 + \frac{1}{4}\frac{b'(0;p)}{b(0;p)} + \frac{1}{2}a(0;p)\Big]y(0;p)$$

and

$$y'(\tau;p) = \frac{1}{\sqrt{b(1;p)}}\Big[1 + \frac{1}{4}\frac{b'(1;p)}{b(1;p)} + \frac{1}{2}a(1;p)\Big]y(1;p).$$

Now one can proceed to the investigation of the existence of approximate solutions as well as to their computation.

References

[1] Donald S. Cohen, Singular Perturbation of Nonlinear Two-Point Boundary-Value Problems, *J. Math. Anal.* **43** (1973), pp. 151-160.

[2] M. Cui, F. Geng, A computational method for solving third-order singularly perturbed boundary-value problems, *Appl. MAth. Comput.* **198** (2008), pp. 896-903.

[3] Zengji Du and Zhanbing Bai, Asymptotic solutions for a second - order differential equation with three-point boundary conditions, *Appl. Math. Comput.* **186** (2007), pp. 469-473.

[2] made with the help of Graphing Calculator 3.5 of Pacific Tech

[4] Zengji Du, Weigao Ge, Mingru Zhou, Singular perturbations for third-order nonlinear multi-point boundary value problem, *J. Differ. Equations* **218** (2005), pp. 69-90.

[5] Dragoslav Herceg, Djordje Herceg, On a fourth-order finite-difference method for singularly perturbed boundary value problems, *Appl. Math. Comput.* **203** (2008), pp. 828-837.

[6] J. Dugundji and A. Granas, *Fixed Point Theory,* Vol. I, Monographie Matematyczne, Warsaw, 1982.

[7] U. Elias and H. Gilgold, Effects of varying nonlinearity and their singular perturbation flavour, *J. Math. Anal. Appl.*, **248**(2000), pp. 309-326.

[8] F. Hoppensteadt, Properties of solutions of ordinary differential equations with small parameters, *Comm. Pure Appl. Math.* 24(1971), pp. 807-840.

[9] Anatoli Ivanov, Eduardo Liz, Periodic solutions of a singular differential delay equation with the Farey-type nonlinearity, *J. Comput. Appl. Math.* **180** (2005), pp. 137-145.

[10] Mohan K. Kadalbajoo, Kailash C. Patidar, A survey of numerical techniques for solving singularly perturbed ordinary differential equations, *Appl. Math. Comput.* **130** (2002), pp. 457-510.

[11] Mohan K. Kadalbajoo, Devendra Kumar, A non-linear single step explicit scheme for non-linear two-point singularly perturbed boundary value problems via initial value technique, *Appl. Math. Comput.* **202** (2008), pp. 738-746.

[12] R. E. O'Malley, Boundary layer methods for nonlinear initial value problems, *SIAM Review,* 13(1971), pp. 425-434.

[13] R. E. O'Malley, On initial value problems for nonlinear systems of differential equations with two small parameters, *Arch. Rational Mech. Anal.* , 40 (1972), pp. 209-222.

[14] R. E. O'Malley, *Singular Perturbation Methods for Ordinary Differential Equations*, Springer, New York, 1991.

[15] Donal O' Regan *Theory of Singular Boundary Value Problems*, World Scientific Publishing Co., Singapore 1994.

[16] Lawrence Perko, *Differential Equations and Dynamical Systems*, Springer-Verlag New York, 1991.

[17] J. I. Ramos, A smooth locally-analytical technique for singularly perturbed two-point boundary-value problems, *Appl. Math. Comput.* **163** (2005), pp. 1123-1142.

[18] Walter Rudin, *Real and Complex Analysis*, TATA McGraw-Hill Publ.Co., 8th Edition, New York, 1983.

[19] D. R. Smith, *Singular perturbation theory*, Cambridge University Press, New York, 1985.

[20] Mirjana Stojanović, A difference method for non-self-adjoint singular perturbation problem of second order, *Appl. Math. Modelling* **15** (1991), pp. 381-385.

[21] Mirjana Stojanović, Global convergence method for singularly perturbed boundary value problems, *J. Comput. Appl. Math.* **181** (2005), pp. 326-335.

[22] Ferdinand Verhust, *Nonlinear Differential equations and Dynamical Systems*, Springer-Verlang, Berlin 1990.

[23] Xiao-Yun Wang, Yao-Lin Jiang, A general method for solving singular perturbed impulsive differential equations with two-point boundary conditions , *Appl. Math. Comput.* **171** (2005), pp. 775-806.

George L. Karakostas
Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
E-mail address: gkarako@uoi.gr

Oscillation criteria for nonlinear neutral hyperbolic equations with functional arguments

Y. SHOUKAKU [1], I. P. STAVROULAKIS [2] AND N. YOSHIDA [3]

Abstract

This paper is devoted to the study of oscillatory behavior of solutions to nonlinear neutral hyperbolic equations with functional arguments by using the integral averaging method and generalized Riccati techniques. First, we establish oscillation results for nonlinear neutral hyperbolic equations by reducing the multi-dimensional oscillation problems to one-dimensional oscillation problems for functional differential inequalities. Secondly, we present oscillation results for nonlinear neutral hyperbolic equations by utilizing Riccati techniques.

Keywords : Oscillation, hyperbolic equations, neutral type, Riccati inequality
2000MSC : 34K11, 35B05, 35R10

1. Introduction

Consider the hyperbolic equation with functional arguments

$$
\text{(E)}\qquad \frac{\partial}{\partial t}\left(r(t)\frac{\partial}{\partial t}\left(u(x,t)+\sum_{i=1}^{l}h_i(t)u(x,\rho_i(t))\right)\right)
-a(t)\Delta u(x,t)-\sum_{i=1}^{k}b_i(t)\Delta u(x,\tau_i(t))
+\sum_{i=1}^{m}q_i(x,t)\varphi_i(u(x,\sigma_i(t)))=0,\ \ (x,t)\in\Omega\equiv G\times(0,\infty),
$$

[1]Faculty of Engineering, Kanazawa University,Kanazawa 920-1192, Japan; e-mail: shoukaku@t.kanazawa-u.ac.jp

[2]Department of Mathematics, University of Ioannina 451-10, Ioannina, Greece; e-mail: ipstav@uoi.gr

[3]Department of Mathematics, University of Toyama, Toyama 930-8555, Japan; e-mail: nori@sci.u-toyama.ac.jp

where Δ is the Laplacian in $\mathbb{R}^n$ and G is a bounded domain of $\mathbb{R}^n$ with piecewise smooth boundary ∂G, and the following Dirichlet and Robin (cf. [10]) boundary conditions:

(B1) $$u = 0 \quad \text{on} \quad \partial G \times [0, \infty),$$

(B2) $$\frac{\partial u}{\partial \nu} + \mu u = 0 \quad \text{on} \quad \partial G \times [0, \infty),$$

where ν denotes the unit exterior normal vector to ∂G and $\mu \in C(\partial G \times [0, \infty); [0, \infty))$.

Throughout this paper we assume that:

A1)]$r(t) \in C^1([0, \infty); (0, \infty))$,
$h_i(t) \in C^2([0, \infty); [0, \infty))$ $(i = 1, 2, \ldots, l)$,
$a(t),\ b_i(t) \in C([0, \infty); [0, \infty))$ $(i = 1, 2, \ldots, k)$,
$q_i(x, t) \in C(\overline{\Omega}; [0, \infty))$ $(i = 1, 2, \ldots, m)$; $\rho_i(t) \in C^2([0, \infty); \mathbb{R})$, $\lim_{t\to\infty} \rho_i(t) = \infty$ $(i = 1, 2, \ldots, l)$,
$\tau_i(t) \in C([0, \infty); \mathbb{R})$, $\lim_{t\to\infty} \tau_i(t) = \infty$ $(i = 1, 2, \ldots, k)$,
$\sigma_i(t) \in C([0, \infty); \mathbb{R})$, $\lim_{t\to\infty} \sigma_i(t) = \infty$ $(i = 1, 2, \ldots, m)$; $\varphi_i(s) \in C^1(\mathbb{R}; \mathbb{R})$ $(i = 1, 2, \ldots, m)$ are convex in $(0, \infty)$ and $\varphi_i(-s) = -\varphi_i(s)$ for $s \geq 0$.

Definition 1. By a *solution* of Eq. (E) we mean a function $u \in C^2(\overline{G} \times [t_{-1}, \infty)) \cap C(\overline{G} \times [\tilde{t}_{-1}, \infty))$ which satisfies (E), where

$$t_{-1} = \min\left\{0, \min_{1\leq i\leq l}\left\{\inf_{t\geq 0} \rho_i(t)\right\}, \min_{1\leq i\leq k}\left\{\inf_{t\geq 0} \tau_i(t)\right\}\right\},$$
$$\tilde{t}_{-1} = \min\left\{0, \min_{1\leq i\leq m}\left\{\inf_{t\geq 0} \sigma_i(t)\right\}\right\}.$$

Definition 2. A solution u of Eq. (E) is said to be *oscillatory* in Ω if u has a zero in $G \times (t, \infty)$ for any $t > 0$.

Definition 3. We say that the functions (H_1, H_2) belong to a function class $\mathcal{H}$, denoted by $(H_1, H_2) \in \mathcal{H}$, if $(H_1, H_2) \in C(D; [0, \infty))$ satisfy

$$H_i(t, t) = 0, \ H_i(t, s) > 0 \ (i = 1, 2) \quad \text{for } t > s,$$

where $D = \{(t, s) : 0 < s \leq t < \infty\}$, and the partial derivatives $\partial H_1/\partial t$ and $\partial H_2/\partial s$ exist on D such that

$$\frac{\partial H_1}{\partial t}(s, t) = h_1(s, t)H_1(s, t) \quad \text{and} \quad \frac{\partial H_2}{\partial s}(t, s) = -h_2(t, s)H_2(t, s),$$

for some functions $h_1,\ h_2 \in C_{loc}(D;\mathbb{R})$, where $C_{loc}(D;\mathbb{R})$ denotes the set of all locally continuous functions on D.

In recent years there has been much research activity concerning the oscillation theory of nonlinear hyperbolic equations with functional arguments by employing Riccati techniques. Riccati techniques were used to obtain various oscillation results (cf. Mařík [9], Yoshida [15]). For example, we note that Kamenev-type oscillation criteria for hyperbolic equations have been obtained in [3,6,12,14]. On the other hand, interval oscillation criteria for second order differential equation have been investigated by many authors [1,3,5,6,8,12,13]. In particular, Wang, Meng and Liu [12,13] applied interval oscillation criteria to linear hyperbolic equations with functional arguments. Recently, Cui and Xu [1] presented oscillation criteria for hyperbolic equations which are not of neutral type. It seems that there are no known oscillation results for hyperbolic equations of neutral type, which are obtained by Riccati techniques.

The objective of this paper is to establish oscillation ceireria for the nonlinear neutral hyperbolic equation with functional arguments (E) by employing the Riccati method.

In Section 2 we reduce our problems to one-dimensional problems for functional differential inequalities, and second order functional differential inequalities are investigated in Section 3 via Riccati inequalities. We present oscillation results for (E) in Section 4 by combining the results of Sections 2 and 3. Two examples which illustrate our main theorems are given in Section 5.

2. Reduction to one-dimensional problems

In this section we reduce the multi-dimensional oscillation problems for (E) to one-dimensional oscillation problems. It is known that the first eigenvalue λ_1 of the eigenvalue problem

$$\begin{aligned} -\Delta w &= \lambda w \quad \text{in} \quad G, \\ w &= 0 \quad \text{on} \quad \partial G \end{aligned}$$

is positive, and the corresponding eigenfunction $\Phi(x)$ can be chosen so that $\Phi(x) > 0$ in G. Now we let

$$q_i(t) = \min_{x \in G} q_i(x,t).$$

With each solution $u(x,t)$ of the problem (E), (B1) or (E), (B2) we associate functions $U(t)$ and $\tilde{U}(t)$ respectively, defined by

$$U(t) = K_\Phi \int_G u(x,t)\Phi(x)dx,$$
$$\tilde{U}(t) = \frac{1}{|G|}\int_G u(x,t)dx,$$

where $K_\Phi = (\int_G \Phi(x)dx)^{-1}$ and $|G| = \int_G dx$.

Theorem 1. *If the functional differential inequality*

$$\frac{d}{dt}\left(r(t)\frac{d}{dt}\left(y(t) + \sum_{i=1}^{l} h_i(t)y(\rho_i(t))\right)\right) + \sum_{i=1}^{m} q_i(t)\varphi_i(y(\sigma_i(t))) \le 0 \qquad (1)$$

has no eventually positive solutions, then every solution $u(x,t)$ of the problem (E), (B1) is oscillatory in Ω.

Proof. Suppose to the contrary that there exists a nonoscillatory solution u of the problem (E), (B1). Without loss of generality we may assume that $u(x,t) > 0$ in $G \times [t_0,\infty)$ for some $t_0 > 0$. (The case where $u(x,t) < 0$ can be treated similarly). Since (A2) holds, we see that $u(x,\rho_i(t)) > 0$ $(i = 1,2,\ldots,l)$, $u(x,\tau_i(t)) > 0$ $(i = 1,2,\ldots,k)$ and $u(x,\sigma_i(t)) > 0$ $(i = 1,2,\ldots,m)$ in $G \times [t_1,\infty)$ for some $t_1 \ge t_0$. Multiplying (E) by $K_\Phi\Phi(x)$ and integrating over G, we obtain

$$\frac{d}{dt}\left(r(t)\frac{d}{dt}\left(U(t) + \sum_{i=1}^{l} h_i(t)U(\rho_i(t))\right)\right)$$
$$-a(t)K_\Phi \int_G \Delta u(x,t)\Phi(x)dx - \sum_{i=1}^{m} b_i(t)K_\Phi \int_G \Delta u(x,\tau_i(t))\Phi(x)dx$$
$$+\sum_{i=1}^{m} K_\Phi \int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))\Phi(x)dx = 0, \; t \ge t_1. \qquad (2)$$

From Green's formula it follows that

$$K_\Phi \int_G \Delta u(x,t)\Phi(x)dx = -\lambda_1 U(t) \le 0, \; t \ge t_1, \qquad (3)$$
$$K_\Phi \int_G \Delta u(x,\tau_i(t))\Phi(x)dx = -\lambda_1 U(\tau_i(t)) \le 0, \; t \ge t_1. \qquad (4)$$

Using the Jensen's inequality we observe that

$$\sum_{i=1}^{m} K_\Phi \int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))\Phi(x)dx \geq \sum_{i=1}^{m} q_i(t)\varphi_i(U(\sigma_i(t))),\ t \geq t_1, \quad (5)$$

and combining (2)–(5), it follows that

$$\frac{d}{dt}\left(r(t)\frac{d}{dt}\left(U(t) + \sum_{i=1}^{l} h_i(t)U(\rho_i(t))\right)\right) + \sum_{i=1}^{m} q_i(t)\varphi_i(U(\sigma_i(t))) \leq 0,\ t \geq t_1.$$

Therefore $U(t)$ is an eventually positive solution of (1). This is a contradiction and the proof is complete.

Theorem 2. *If the functional differential inequality (1) has no eventually positive solutions, then every solution $u(x,t)$ of the problem (E), (B2) is oscillatory in Ω.*

Proof. Suppose to the contrary that there exists a nonoscillatory solution u of the problem (E), (B2). Without loss of generality we may assume that $u(x,t) > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$. Since (A2) holds, we see that $u(x, \rho_i(t)) > 0$ $(i = 1, 2, \dots, l)$, $u(x, \tau_i(t)) > 0$ $(i = 1, 2, \dots, k)$ and $u(x, \sigma_i(t)) > 0$ $(i = 1, 2, \dots, m)$ in $G \times [t_1, \infty)$ for some $t_1 \geq t_0$. Dividing (E) by $|G|$ and integrating over G, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(r(t)\frac{d}{dt}\left(\tilde{U}(t) + \sum_{i=1}^{l} h_i(t)\tilde{U}(\rho_i(t))\right)\right) \\
&-\frac{a(t)}{|G|}\int_G \Delta u(x,t)dx - \sum_{i=1}^{k} \frac{b_i(t)}{|G|}\int_G \Delta u(x,\tau_i(t))dx \\
&+\frac{1}{|G|}\sum_{i=1}^{m}\int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))dx = 0,\ t \geq t_1. \qquad (6)
\end{aligned}$$

It follows from Green's formula that

$$\begin{aligned}\int_G \Delta u(x,t)dx &= \int_{\partial G}\frac{\partial u}{\partial \nu}(x,t)dS \\ &= -\int_{\partial G}\mu(x,t)u(x,t)dS \le 0,\ t\ge t_1, \qquad (7)\\ \int_G \Delta u(x,\tau_i(t))dx &= \int_{\partial G}\frac{\partial u}{\partial \nu}(x,\tau_i(t))dS \\ &= -\int_{\partial G}\mu(x,\tau_i(t))u(x,\tau_i(t))dS \le 0,\ t\ge t_1. \qquad (8)\end{aligned}$$

Using the Jensen's inequality, we observe that

$$\sum_{i=1}^{m} K_\Phi \int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))dx \ge \sum_{i=1}^{m} q_i(t)\varphi_i(\tilde{U}(\sigma_i(t))),\ t\ge t_1, \qquad (9)$$

and combining (6)–(9), it follows that

$$\frac{d}{dt}\left(r(t)\frac{d}{dt}\left(\tilde{U}(t)+\sum_{i=1}^{l} h_i(t)\tilde{U}(\rho_i(t))\right)\right)+\sum_{i=1}^{m} q_i(t)\varphi_i(\tilde{U}(\sigma_i(t)))\le 0,\ t\ge t_1.$$

Therefore $\tilde{U}(t)$ is an eventually positive solution of (1). This is a contradiction and the proof is complete.

3. Second order functional differential inequalities

In this section we establish sufficient conditions for every solution $y(t)$ of the functional differential inequality (1) to have no eventually positive solution. We assume the following hypotheses:

(A1)(A2)(A3) For some $j \in \{1,2,\ldots,m\}$, there exists a positive constants σ such that

$$\sigma_j'(t)\ge\sigma \quad \text{and} \quad \sigma_j(t)\le t,$$

and $\varphi_j(s)\in C^1((0,\infty);(0,\infty))$, $\varphi_j'(s)>0$ and $\varphi_j'(s)$ is nondecreasing for $s>0$;

$$\int_{t_0}^{\infty}\frac{1}{r(t)}dt=\infty;$$

$\sum_{i=1}^{l} h_i(t) \leq 1;$

$\rho_i(t) \leq t \ (i = 1, 2, \ldots, l);$

Theorem 3. *Assume that the hypotheses (A4)–(A7) hold, and moreover assume that*

$\varphi_j(s_1 s_2) \geq \varphi_{j1}(s_1)\varphi_{j2}(s_2)$ for $s_1 \geq 0$, $s_2 > 0$, where $\varphi_{j1}(s) \in C([0,\infty); [0,\infty))$, $\varphi_{j2}(s) \in C^1((0,\infty); (0,\infty))$ and $\varphi_{j2}(s)$ is nondecreasing for $s > 0$.

If the Riccati inequality

$$z'(t) + \frac{1}{2}\frac{1}{P_{\tilde{K}}(t)} z^2(t) \leq -Q(t) \tag{10}$$

for some $\tilde{K} > 0$ and all large T, has no solution on $[T, \infty)$, where

$$P_{\tilde{K}}(t) = \frac{r(\sigma_j(t))}{2\tilde{K}\sigma}, \tag{11}$$

$$Q(t) = q_j(t)\varphi_{j1}\left(1 - \sum_{i=1}^{l} h_i(\sigma_j(t))\right), \tag{12}$$

then (1) has no eventually positive solutions.

Proof. Suppose that $y(t)$ is a positive solution of (1) on $[t_0, \infty)$ for some $t_0 > 0$. From (1), there exists a $j \in \{1, 2, \ldots, m\}$ such that

$$\frac{d}{dt}\left(r(t)\frac{d}{dt}\left(y(t) + \sum_{i=1}^{l} h_i(t) y(\rho_i(t))\right)\right) + q_j(t)\varphi_j(y(\sigma_j(t))) \leq 0, \ t \geq t_0.$$

If we define the function

$$z(t) = y(t) + \sum_{i=1}^{l} h_i(t) y(\rho_i(t)), \tag{13}$$

then we see that

$$(r(t)z'(t))' \leq -q_j(t)\varphi_j(y(\sigma_j(t))) \leq 0, \ t \geq t_0. \tag{14}$$

Since $(r(t)z'(t))' \leq 0$, $z(t) > 0$ eventually, we observe, using the hypothesis (A5), that $z'(t) \geq 0$ $(t \geq t_1)$ for some $t_1 > t_0$ (cf. [13, Lemma 2.2]). Hence $r(t)z'(t)$ is nonincreasing. Then, we find that $z'(t) \geq 0$ or $z'(t) < 0$ for

$t \geq t_1 > t_0$. First we assume that $z'(t) < 0$ for $t \geq t_1$. From the well known argument (cf. [13]) we prove that $z'(t) \geq 0$ for $t \geq t_1$. Taking into account (A6) and (A7), from (13) we see that (cf. Yoshida [15])

$$y(t) \geq \left(1 - \sum_{i=1}^{l} h_i(t)\right) z(t), \ t \geq t_1. \tag{15}$$

In view of (14) and (15), we observe that

$$(r(t)z'(t))' + q_j(t)\varphi_{j1}\left(1 - \sum_{i=1}^{l} h_i(\sigma_j(t))\right) \varphi_{j2}(z(\sigma_j(t))) \leq 0, \ t \geq t_1.$$

Setting

$$w(t) = \frac{r(t)z'(t)}{\varphi_{j2}(z(\sigma_j(t)))},$$

we show that

$$w'(t) = \frac{(r(t)z'(t))'}{\varphi_{j2}(z(\sigma_j(t)))} - r(t)z'(t)\frac{\varphi'_{j2}(z(\sigma_j(t)))z'(\sigma_j(t))\sigma'_j(t)}{\varphi^2_{j2}(z(\sigma_j(t)))}. \tag{16}$$

Since $z(t) > 0$, $z'(t) \geq 0$ eventually, it follows that $z(\sigma_j(t)) \geq k_0$ for some $k_0 > 0$. Hence we observe that

$$\varphi'_{j2}(z(\sigma_j(t))) \geq \varphi'_{j2}(k_0) \equiv \tilde{K}. \tag{17}$$

Substituting (17) into (16), we get

$$\begin{aligned} w'(t) \ &\leq \ -q_j(t)\varphi_{j1}\left(1 - \sum_{i=1}^{l} h_i(\sigma_j(t))\right) \\ &\qquad -\tilde{K}\sigma r(t)z'(t)\frac{z'(\sigma_j(t))}{\varphi^2_{j2}(z(\sigma_j(t)))}, \ t \geq t_1. \end{aligned}$$

On the other hand, (14) implies that

$$r(\sigma_j(t))z'(\sigma_j(t)) \geq r(t)z'(t),$$

and hence

$$w'(t) + \frac{1}{2}\left(\frac{2\tilde{K}\sigma}{r(\sigma_j(t))}\right) w^2(t) \leq -q_j(t)\varphi_{j1}\left(1 - \sum_{i=1}^{l} h_i(\sigma_j(t))\right) \tag{18}$$

for $t \geq t_1$. That is, $w(t)$ is a solution of (10) on $[t_1, \infty)$. This is a contradiction and the proof is complete.

Theorem 4. *Assume that the hypotheses (A4)–(A8) hold. If for each $T > 0$ and some $\tilde{K} > 0$, there exist $(H_1, H_2) \in \mathcal{H}$, $\psi(t) \in C^1((0,\infty);(0,\infty))$ and $a, b, c \in \mathbb{R}$ such that $T \leq a < c < b$ and*

$$\frac{1}{H_1(c,a)} \int_a^c H_1(s,a) \left\{ Q(s) - \frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma} \lambda_1^2(s,a) \right\} \psi(s) ds$$
$$+ \frac{1}{H_2(b,c)} \int_c^b H_2(b,s) \left\{ Q(s) - \frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma} \lambda_2^2(b,s) \right\} \psi(s) ds > 0, \quad (19)$$

where

$$\begin{aligned} \lambda_1(s,t) &= \frac{\psi'(s)}{\psi(s)} + h_1(s,t), \\ \lambda_2(t,s) &= \frac{\psi'(s)}{\psi(s)} - h_2(t,s). \end{aligned}$$

Then (1) has no eventually positive solutions.

Proof. Suppose that $y(t)$ is a positive solution of (1) on $[t_0, \infty)$ for some $t_0 > 0$. At first, we assume that $y(t) > 0$ on (a, b). Proceeding as in the proof of Theorem 3, we see that there exists a function $w(s)$ which satisfies

$$Q(s)\psi(s) \leq -w'(s)\psi(s) - \frac{\tilde{K}\sigma}{r(\sigma_j(s))} w^2(s)\psi(s). \quad (20)$$

Multiplying (20) by $H_2(t,s)$ and integrating over $[c,t]$ for $t \in [c,b)$, we have

$$\begin{aligned} &\int_c^t H_2(t,s)Q(s)\psi(s)ds \\ \leq\ & -\int_c^t H_2(t,s)w'(s)\psi(s)ds - \int_c^t H_2(t,s)\frac{\tilde{K}\sigma}{r(\sigma_j(s))} w^2(s)\psi(s)ds \\ \leq\ & H_2(t,c)w(c)\psi(c) + \frac{1}{4}\int_c^t H_2(t,s)\lambda_2^2(t,s)\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\psi(s)ds \\ & -\int_c^t H_2(t,s)\left\{ \sqrt{\frac{\tilde{K}\sigma}{r(\sigma_j(s))}} w(s) - \frac{1}{2}\lambda_2(t,s)\sqrt{\frac{r(\sigma_j(s))}{\tilde{K}\sigma}} \right\}^2 \psi(s)ds, \end{aligned}$$

and so

$$\frac{1}{H_2(t,c)}\int_c^t H_2(t,s)\left\{Q(s)-\frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\lambda_2^2(t,s)\right\}\psi(s)ds \le w(c)\psi(c).$$

Letting $t \to b^-$ in the last inequality, we obtain

$$\frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\left\{Q(s)-\frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\lambda_2^2(b,s)\right\}\psi(s)ds \le w(c)\psi(c). \quad (21)$$

On the other hand, multiplying (20) by $H_1(s,t)$ and integrating over $[t,c]$ for $t \in (a,c]$, we obtain

$$\begin{aligned}
&\int_t^c H_1(s,t)q_j(s)\psi(s)ds \\
\le\ & -\int_t^c H_1(s,t)w'(s)\psi(s)ds - \int_t^c H_1(s,t)\frac{\tilde{K}\sigma}{r(\sigma_j(s))}w^2(s)\psi(s)ds \\
\le\ & -H_1(c,t)w(c)\psi(c) + \frac{1}{4}\int_t^c H_1(s,t)\lambda_1^2(s,t)\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\psi(s)ds \\
& -\int_t^c H_1(s,t)\left\{\sqrt{\frac{\tilde{K}\sigma}{r(\sigma_j(s))}}w(s) - \frac{1}{2}\lambda_1(s,t)\sqrt{\frac{r(\sigma_j(s))}{\tilde{K}\sigma}}\right\}^2\psi(s)ds,
\end{aligned}$$

and therefore

$$\frac{1}{H_1(c,t)}\int_t^c H_1(s,t)\left\{Q(s)-\frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\lambda_1^2(s,t)\right\}\psi(s)ds \le -w(c)\psi(c).$$

Letting $t \to a^+$ in the last inequality, we obtain

$$\frac{1}{H_1(c,a)}\int_a^c H_1(s,a)\left\{Q(s)-\frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\lambda_1^2(s,a)\right\}\psi(s)ds \le -w(c)\psi(c). \quad (22)$$

Adding (21) and (22), we obtain the following

$$\begin{aligned}
&\frac{1}{H_1(c,a)}\int_a^c H_1(s,a)\left\{Q(s)-\frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\lambda_1^2(s,a)\right\}ds \\
&\quad+\frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\left\{Q(s)-\frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\lambda_2^2(b,s)\right\}ds \le 0,
\end{aligned}$$

which contradicts the condition (19). Pick up a sequence $\{T_i\} \subset [t_0, \infty)$ such that $T_i \to \infty$ as $i \to \infty$. By the assumptions, for each $i \in \mathbb{N}$, there exists a_i, b_i, $c_i \in [0, \infty)$ such that $T_i \le a_i < c_i < b_i$, and (19) holds with a, b, c replaced by a_i, b_i, c_i, respectively. Therefore, every nontrivial solution $y(t)$ of (1) has at least one zero $t_i \in (a_i, b_i)$. Noting that $t_i > a_i \ge T_i$, $i \in \mathbb{N}$, we see that $y(t)$ is an oscillatory solution of (1). This is a contradiction and the proof is complete.

Theorem 5. *Assume that the hypotheses (A4)–(A8) hold. If for each $T > 0$ and some $\tilde{K} > 0$, there exist functions $(H_1, H_2) \in \mathcal{H}$, $\psi(t) \in C^1((0,\infty);(0,\infty))$, such that*

$$\limsup_{t\to\infty} \int_T^t H_1(s,T)\left\{Q(s) - \frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\lambda_1^2(s,T)\right\}\psi(s)ds > 0 \tag{23}$$

and

$$\limsup_{t\to\infty} \int_T^t H_2(t,s)\left\{Q(s) - \frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\lambda_2^2(t,s)\right\}\psi(s)ds > 0, \tag{24}$$

then (1) has no eventually positive solutions.

Proof. For any $T \ge t_0$, let $a = T$ and choose $T = a$ in (23). Then there exists $c > a$ such that

$$\int_a^c H_1(s,a)\left\{Q(s) - \frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\lambda_1^2(s,a)\right\}\psi(s)ds > 0. \tag{25}$$

Next, choose $T = c$ in (24). Then there exists $b > c$ such that

$$\int_c^b H_2(b,s)\left\{Q(s) - \frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\lambda_2^2(b,s)\right\}\psi(s)ds > 0. \tag{26}$$

Combining (25) and (26), we obtain (19). By the virtue of Theorem 4, the proof is complete.

4. Oscillation criteria for Eq. (E)

In this section, by combining the results of Sections 2 and 3, we establish sufficient conditions for oscillation of Eq. (E).

Using the Riccati inequality, we derive sufficient conditions for every solution of hyperbolic equation (E) to be oscillatory. We are going to use the following lemma which is due to Usami [11].

Lemma. *If there exists a function* $\psi(t) \in C^1([T_0,\infty);(0,\infty))$ *such that*

$$\int_{T_1}^{\infty} \left(\frac{\bar{p}(t)|\psi'^{\beta}}{\psi(t)}\right)^{\frac{1}{\beta-1}} dt < \infty,$$
$$\int_{T_1}^{\infty} \frac{1}{\bar{p}(t)(\psi(t))^{\beta-1}} dt = \infty,$$
$$\int_{T_1}^{\infty} \psi(t)\bar{q}(t)dt = \infty$$

for some $T_1 \geq T_0$, *then the Riccati inequality*

$$x'(t) + \frac{1}{\beta}\frac{1}{\bar{p}(t)}|x(t)|^{\beta} \leq -\bar{q}(t),$$

where $\beta > 1$, $\bar{p}(t) \in C([T_0,\infty);(0,\infty))$ *and* $\bar{q}(t) \in C([T_0,\infty);\mathbb{R})$, *has no solution on* $[T,\infty)$ *for all large* T.

Combining Theorems 1-3 and Lemma, we obtain the following theorem.

Theorem 6. *Assume that the hypotheses (A1)–(A7) hold. If*

$$\int_{T_1}^{\infty} \left(\frac{P_{\tilde{K}}(t)\psi'^2}{\psi(t)}\right) dt < \infty,$$
$$\int_{T_1}^{\infty} \frac{1}{P_{\tilde{K}}(t)\psi(t)} dt = \infty,$$
$$\int_{T_1}^{\infty} \psi(t)Q(t)dt = \infty,$$

where $P_{\tilde{K}}(t)$ *and* $Q(t)$ *are defined by (11) and (12) for some* $\tilde{K} > 0$, *then every solution* $u(x,t)$ *of (E), (B1) (or (E), (B2)) is oscillatory in* Ω.

Combining Theorems 1–2 and 4, we have the following theorem.

Theorem 7. *Assume that the hypotheses (A1)–(A7) hold. If for each* $T > 0$ *and some* $\tilde{K} > 0$, *there exist functions* $(H_1,H_2) \in \mathcal{H}$, $\psi(t) \in C^1((0,\infty);(0,\infty))$ *and* $a,b,c \in \mathbb{R}$ *such that* $T \leq a < c < b$ *and (19) hold, then every solution* $u(x,t)$ *of (E), (B1) (or (E), (B2)) is oscillatory in* Ω.

Analogously, combining Theorems 1–2 and 5 we derive the following.

Theorem 8. *Assume that the hypotheses (A1)–(A7) hold. If for each* $T > 0$ *and some* $\tilde{K} > 0$, *there exist functions* $(H_1,H_2) \in \mathcal{H}$, $\psi(t) \in$

$C^1((0,\infty);(0,\infty))$ such that (23) and (24) hold, then every solution $u(x,t)$ of (E), (B1) (or (E), (B2)) is oscillatory in Ω.

5. Examples

We present the following examples which illustrate the applicability of our results.

Example 1. Consider the problem

$$\begin{aligned} &\frac{\partial}{\partial t}\left(e^{-t}\frac{\partial}{\partial t}\left(u(x,t)+\frac{1}{2}u(x,t-\pi)\right)\right)-\frac{1}{2}e^{-t}\Delta u(x,t)\\ &\quad -\frac{1}{2}e^{-t}\Delta u\left(x,t+\frac{\pi}{2}\right)-e^{2t}\Delta u(x,t-2\pi)\\ &\quad +e^{2t}u(x,t-\pi)=0,\ (x,t)\in(0,\pi)\times[1,\infty), \qquad (27)\\ &u(0,t)=u(\pi,t)=0. \qquad (28)\end{aligned}$$

Here $n=1$, $k=2$, $m=1$, $r(t)=e^{-t}$, $h_1(t)=1/2$, $q_1(x,t)=e^{2t}$, $\sigma_1(t)=t-\pi$ and $\varphi'_{12}(\xi)=1=\tilde{K}$. It is easy to see that

$$P_{\tilde{K}}(t)=\frac{1}{2}e^{-t+\pi},\quad Q(t)=\frac{1}{2}e^{2t}.$$

By choosing

$$\psi(t)=e^{-2t},\quad H_1(s,t)=H_2(t,s)=\left(e^t-e^s\right)^2,$$

we see that

$$\begin{aligned}&\int^{\infty}\left(\frac{\frac{1}{2}e^{-t+\pi}(-2e^{-2t})^2}{e^{-2t}}\right)dt=\int^{\infty}2e^{-3t+\pi}dt<\infty,\\ &\int^{\infty}\left(\frac{1}{\frac{1}{2}e^{-t+\pi}\times e^{-2t}}\right)dt=\int^{\infty}2e^{3t-\pi}dt=\infty,\\ &\int^{\infty}\left(e^{-2t}\times\frac{1}{2}e^{2t}\right)dt=\infty.\end{aligned}$$

Choose now $a=0$, $b=2\pi$ and $c=\pi$ and observe that

$$\begin{aligned}&\frac{1}{(1-e^{\pi})^2}\int_0^{\pi}(1-e^s)^2\left\{\frac{1}{2}e^{2s}-\frac{1}{4}e^{-s+\pi}\frac{4e^{2s}}{(e^s-1)^2}\right\}e^{-2s}ds\\ &+\frac{1}{(e^{2\pi}-e^{\pi})^2}\int_{\pi}^{2\pi}(e^{2\pi}-e^s)^2\left\{\frac{1}{2}e^{2s}-\frac{1}{4}e^{-s+\pi}\frac{4e^{4\pi}}{(e^{2\pi}-e^s)^2}\right\}e^{-2s}ds>0,\end{aligned}$$

that is, the condition (19) is satisfied. Also

$$\limsup_{t\to\infty}\int_T^t (e^s - s^T)^2 \left\{\frac{1}{2}e^{2s} - \frac{1}{4}e^{-s+\pi}\frac{4e^{2s}}{(e^s-e^T)^2}\right\} e^{-2s}ds$$
$$= \limsup_{t\to\infty}\left\{\frac{1}{4}e^{2t} - e^{t+T} + \frac{1}{2}\left(t - T + \frac{3}{2}\right)e^{2T} + e^{-t+\pi} - e^{-T+\pi}\right\} > 0$$

and

$$\limsup_{t\to\infty}\int_T^t (e^t - s^s)^2 \left\{\frac{1}{2}e^{2s} - \frac{1}{4}e^{-s+\pi}\frac{4e^{2t}}{(e^t-e^s)^2}\right\} e^{-2s}ds$$
$$= \limsup_{t\to\infty}\left\{\left(\frac{1}{2}\left(t - T - \frac{3}{2}\right) - \frac{1}{3}e^{\pi-3T}\right)e^{2t} + e^{t+T} + \frac{1}{3}e^{-t+\pi} - \frac{1}{4}e^{2T}\right\} > 0.$$

that is, the conditions (23) and (24) hold. Thus, all the conditions of Theorems 6–8 are satisfied. Therefore every solution $u(x,t)$ of the problem (27), (28) is oscillatory in $(0,\infty)\times[1,\infty)$. For example, $u(x,t) = \sin x \sin t$ is such a solution.

Example 2. Consider the problem

$$\frac{\partial}{\partial t}\left(\frac{1}{(t+\pi)^2}\frac{\partial}{\partial t}\left(u(x,t) + \frac{1}{2}u(x,t-2\pi)\right)\right) - \Delta u(x,t)$$
$$-\frac{3}{2(t+\pi)^2}\Delta u(x,t-2\pi) - \frac{3}{(t+\pi)^3}\Delta u\left(x, t+\frac{\pi}{2}\right)$$
$$+u(x,t-\pi) = 0,\ (x,t)\in(0,\pi)\times[1,\infty), \tag{29}$$
$$-u_x(0,t) = u_x(\pi,t) = 0. \tag{30}$$

Here $n = 1$, $k = 2$, $m = 1$, $r(t) = (t+\pi)^{-2}$, $h_1(t) = 1/2$, $q_1(x,t) = 1$, $\sigma_1(t) = t - 2\pi$ and $\varphi'_{12}(\xi) = 1 = \tilde{K}$. It is easy to see that

$$P_{\tilde{K}}(t) = \frac{1}{2t^2},\quad Q(t) = \frac{1}{2}.$$

If we choose $\psi(t) = t^2$, then

$$\int^\infty \left(\frac{\frac{1}{2t^2}(2t)^2}{t^2}\right)dt = \int^\infty \frac{2}{t^2}dt < \infty,$$
$$\int^\infty \left(\frac{1}{\left(\frac{1}{2t^2}\right)\times t^2}\right)dt = \infty,$$
$$\int^\infty \left(\frac{1}{2}t^2\right)dt = \infty.$$

Next, choose $\psi(t) = 1$, $H_1(s,t) = H_2(t,s) = (t-s)^2$, and $a = 0$, $b = 2\pi$, $c = \pi$. It is easy to see that

$$\frac{1}{\pi^2}\int_0^{\pi} s^2 \left\{\frac{1}{2} - \frac{1}{4s^2}\frac{4}{s^2}\right\} s^2 ds$$
$$+\frac{1}{\pi^2}\int_{\pi}^{2\pi} (2\pi - s)^2 \left\{\frac{1}{2} - \frac{1}{4s^2}\frac{4}{(2\pi - s)^2}\right\} s^2 ds > 0.$$

Moreover,

$$\limsup_{t\to\infty} \int_T^t (s-T)^2 \left\{\frac{1}{2} - \frac{1}{4}\frac{1}{s^2}\frac{4}{(s-T)^2}\right\} s^2 ds$$
$$= \limsup_{t\to\infty} \left\{\frac{1}{10}t^5 - \frac{1}{4}Tt^4 + \frac{1}{6}T^2t^3 - t - \frac{1}{60}T^5 + T\right\} > 0$$

and

$$\limsup_{t\to\infty} \int_T^t (t-s)^2 \left\{\frac{1}{2} - \frac{1}{4}\frac{1}{s^2}\frac{4}{(t-s)^2}\right\} s^2 ds$$
$$= \limsup_{t\to\infty} \left\{\frac{1}{60}t^5 - \frac{1}{6}T^3t^2 + \left(\frac{1}{4}T^4 - 1\right) t - \frac{1}{10}T^5 + T\right\} > 0.$$

Thus, all the conditions of Theorems 6-8 are satisfied. Therefore, every solution $u(x,t)$ of the problem (29), (30) is oscillatory in $(0,\pi) \times [1,\infty)$. One such solution is $u(x,t) = \cos x \sin t$.

Observe, however, that

$$\int^{\infty} \frac{1}{2}\left(\frac{3}{2(s+\pi)^2} + \frac{3}{(s+\pi)^3}\right) ds < \infty,$$

and therefore the condition (8) of Theorem 2 given by Deng [2] is not satisfied. Thus, Theorem 2 by Deng [2] can not be applied to this example.

References

[1] S. Cui, Z. Xu, Interval oscillation theorems for second order nonlinear partial delay differential equations, Differ. Equ. Appl., 1 (2009) 379–391.

[2] L. Deng, Oscillation criteria for certain hyperbolic functional differential equations with Robin boundary condition, Indian J. Pure Appl. Math., 33 (2002), 1137–1146.

[3] Q. Kong, Interval criteria for oscillation of second-order linear ordinary differential equations, J. Math. Anal. Appl., 229 (1999) 258–270.

[4] W. N. Li, Oscillation for solutions of partial differential equations with delays, Demonstratio Math., 33 (2000) 319–332.

[5] W. T. Li, Interval oscillation criteria for second order nonlinear differential equations with damping, Taiwanese J. Math., 7 (2003) 461–475.

[6] W. T. Li, R. P. Agarwal, Interval oscillation criteria related to integral averaging technique for certain nonlinear differential equations, J. Math. Anal. Appl., 245 (2000) 171–188.

[7] W. N. Li, B. T. Cui, Oscillation of solutions of neutral partial functional differential equations, J. Math. Anal. Appl., 234 (1999) 123–146.

[8] W. T. Li, H. F. Huo, Interval oscillation criteria for nonlinear second-order differential equations, Indian J. Pure appl. Math., 32 (2001) 1003–1014.

[9] R. Mařík, Oscillation Theory of Partial Differential Equations with p-Laplacian, Folia Univ. Agric. et Silvic. Mendel. Brun., Brno, 2008.

[10] I.P. Stavroulakis and S.A. Tersian, Partial Differential Equations, An Introduction with Mathematica and MAPLE (Second Edition), World Scientific Publishing Co., 2004.

[11] H. Usami, Some oscillation theorem for a class of quasilinear elliptic equations, Ann. Mat. Pura Appl., 175 (1998) 277–283.

[12] J. Wang, F. Meng, S. Liu, Integral average method for oscillation of second order partial differential equations with delays, Appl. Math. Comput., 187 (2007) 815–823.

[13] J. Wang, F. Meng, S. Liu, Interval oscillation criteria for second order partial differential equations with delays, J. Comput. Appl. Math., 212 (2008) 397–405.

[14] Q. Yang, On the oscillation of certain nonlinear neutral partial differential equations, Appl. Math. Letters, 20 (2007) 900–907.

[15] N. Yoshida, Oscillation Theory of Partial Differential Equations, World Scientific Publishing Co. Pte. Ltd., 2008.

[16] Y. H. Zhong, Y. H. Yuan, Oscillation criteria of hyperbolic equations with continuous deviating arguments, Kyungpook Math. J., 47 (2007) 347–356.